A Note on the E-mail Game
-
Bounded Rationality and Induction

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Comments welcome

Abstract

In Rubinstein´s (1989) E-mail game there exists no Nash equilibrium where players use strategies that condition on the E-mail communication. In this paper I restrict the utilizable information for one player. I show that in contrast to Rubinstein´s result, in a payoff dominant Nash equilibrium players use strategies that condition on the number of messages sent. Therefore - induction under the assumption of bounded rational behavior of at least one player leads to a more intuitive equilibrium in the E-mail game.

Keywords: Induction, Subgame Perfect Equilibrium, Information sets, Imperfect recall

JEL Classification: C72

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†I am grateful for helpful comments by Jörg Oechssler, Ulrich Kamecke, Elmar Wolfstetter and seminar participants at University College London. Financial support by the Deutsche Forschungsgesellschaft (DFG) through SFB 373 is gratefully acknowledged.

1 Introduction

In his Electronic Mail game Rubinstein (1989)[1] illustrates the difference between common knowledge and "almost common knowledge". Using his example I illustrate another puzzling effect on the equilibrium behavior of this game by applying a notion of imperfect recall to the model. I show that bounded rational behavior in this game almost reestablishes the equilibrium that exists under common knowledge and full rationality.

In the Electronic Mail game two players either play a game G_a (with probability $(1-p) > \frac{1}{2}$) or G_b (with probability $p < \frac{1}{2}$). In each game players choose between action A and B. In both games it is mutually beneficial for players to choose the same action. Figure 1 describes the game. In game a (b) the Pareto dominant equilibrium is the one where players coordinate on A (B). If players chose different actions the player who played B is punished by $-L$ regardless of the game played. The other player gets 0. It is assumed that the potential loss L is not less than the gain M and both are positive.

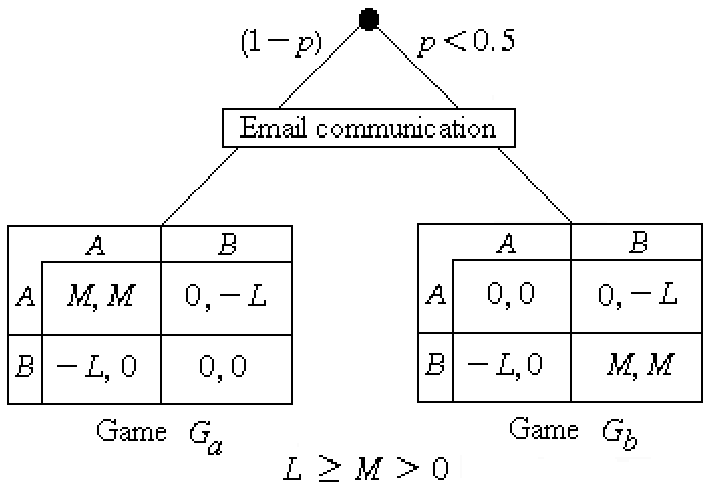

Figure 1: The Email Game

Only player 1 is informed about the game that is actually played. After the state of the world is determined two machines (one for each player) communicate about the game. If game b prevails, player 1´s machine sends an

[1] Osborne and Rubinstein (1994) contains a textbook presentation of the problem.

E-mail message (a beep) to player 2´s machine which is automatically confirmed. This confirmation is confirmed and so on. With a "small" probability ε a message gets lost. Communication stops, when one of the messages (the original message or one of the confirmations) is lost. Players are informed how many messages their machine sent to the other player. Then they have to make their decision.

The Electronic Mail game represents a slight deviation from common knowledge ("almost common knowledge" in Rubinstein´s terms). Combined with perfect rationality this leads to discontinuous drop in expected payoffs. Paradoxically in this case the game has an equilibrium, where players never play the payoff dominant equilibrium in one game (b) even if many messages were sent.

The point I make is that by reducing the ability to process information the existence of an additional subgame perfect equilibrium is guaranteed. The extension I propose is that a player cannot distinguish among the elements in a certain set of numbers, i.e. he cannot distinguish wether $T, T+1, ..., T+l$ messages were sent. If a sufficient number of messages is sent, players in this new equilibrium coordinate on the payoff dominant equilibrium in both games and therefore that equilibrium Pareto dominates an equilibrium where players do not play the payoff dominant equilibrium.

As in related work by Dulleck and Oechssler (1996) the E-mail game is an example where induction under bounded rationality leads to different results. Therefore the hypothesis implied by experimental data that agents do not use induction correctly[2], may be due to the fact that they face limitations on utilizable information which are due to bounded rationality. The E-mail game shows that agents might use induction correctly but in a different environment.

One further result follows from the main results of the paper:

Given the following descending order of the quality of the informational structure: "common knowledge", "almost common knowledge", "almost common knowledge and non-distinguishability", and "no knowledge at all" the expected payoff of the equilibrium under the different regimes vary non-monotonically. This is in contrast to results presented in the economic literature where either knowing less about a characteristic of the state of the world is an advantage but then knowing even less usually does not worsen the

[2] McKelvey and Palfrey (1992) and Rosenthal (1981) among others present experiments on the centipede game that imply this hypothesis.

outcome for a player. Or in other cases, knowing more is better but usually knowing even more does not worsen the result. Note the additional reduction I propose is in the same "dimension" as the reduction in Rubinstein´s original contribution.

The proposed argument can also be applied to solve the related paradox of the *Coordinated Attack* problem (see e.g. in Fagin et al (1995), Chapter 6).

2 The Electronic Mail Game and its extension

Using the notation of Rubinstein (1989) the feasible states s of the world are represented as a triple consisting of the game actually played and the number of messages sent by player 1 and by player 2, i.e. $s \in \{(a,0,0);(b,1,0);(b,1,1);(b,2,1);(b,2,2),...(b,T_1,T_2)...\}$. T_1 and T_2 are the numbers observed by player 1 and player 2 respectively, $T_2 \in \{T_1-1;T_1\}$. For simplicity of notation I will only use a pair consisting of the numbers of messages sent. We must be in game b if and only if $T_1 \geq 1$. Hereby we rule out that the machine of player 1 fails to send a message although we are in state b. Note however that we do not rule out that this message gets lost.

Figure 2 gives a graphical representation of this game, where the automatic moves (by nature) of the machines are represented. The "outcomes" are the numbers players observe before making their decisions.

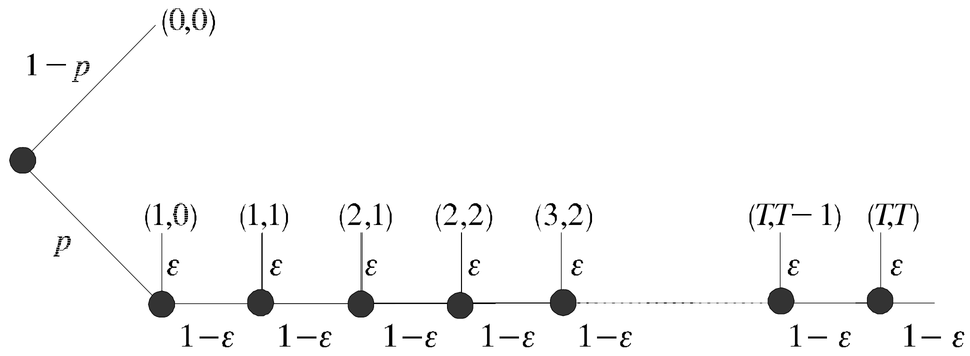

Figure 2: Information "Outcomes" of the email game

In Rubinstein´s game, player 1 cannot distinguish between the "outcomes" $(T_1, T_1 - 1)$ and (T_1, T_1) (and player 2 cannot distinguish between the states (T_2, T_2) and $(T_2 + 1, T_2)$). In this case he always only observes T_1 (T_2). A player chooses his strategy conditioned on the number of messages sent by his machine. A strategy will be played in two states of the world - the two states were the player i observes that T_i messages were sent by his machine. He has to build beliefs about which informational "outcome" is the actual one.

The **extend** the E-mail game by adding non-distinguishability of numbers. In the extended setup one player is not able to distinguish the numbers $t \in \{T, T+1, ..., T+l\}$ where $l \in \mathbb{N}$. This information structure is common knowledge. I refer to this version as the extended game. Otherwise the players play the game as it is described above.

The non-distinguishability represents the case where a player cannot observe or interpret the information about the number of messages sent if they belong to the interval $[T; T+l]$. This might be due to the fact that he is not able to distinguish the numbers (interpret the numbers in the right way) or that the machine is not able to show different symbols if t is in the critical interval.

This modification seems to be obvious given $l \to \infty$, which is the case where the machine or the player lose track at stage T. Justifications for this assumption could be the "overflow" of the machine´s capacities (it can only count up to a certain number) or that real players actually stop counting after they sent a certain number of messages. The result in this case is identical to Rubinstein´s (1989) problem where the maximum number of messages to be sent is limited. I show that the weaker condition that players cannot distinguish between some states is enough to yield a subgame perfect equilibrium with coordination. This weaker condition may be due to minor problems in the processing of information, e.g.. a player can only observe even numbers.

Language differences may be a reason why a player cannot distinguish between, let us say, 17 and 18 (e.g.. he maybe unsure of the right order of 17 and 18)[3]. Or the machine may not be able to show 18 and therefore it stays on 17 for two turns and then jumps to 19.[4]

[3]Assume that one plays the Email game in China using traditional chinese numbers (which were taught before) - I am sure one would get confused interpreting the symbols.

[4]The proposed logic can also be applied to a situation where one or both players count eg. only even numbers. Necessary for the present results is that the information sets

Given this modification of non-distinguishability one has a problem which analysis is similar to that of the problem of imperfect recall[5] in the sense that a player forgets how many beeps he has heard or messaged he received before but he is reminded once in a while about the actual number. The player cannot distinguish/remember whether his machine sent $T, T+1, ...$ or $T+l$ messages and therefore he has to choose one action for all observations in the interval.

3 Results

In the original game, Rubinstein (1989) proves that there is no Nash equilibrium where players condition on the number of messages sent:

Proposition 1 (Rubinstein (1989)) *There is only one Nash equilibrium in which player 1 plays A in game G_a. In this equilibrium players play A independently of the number of messages sent.*

The formal proof is provided in Rubinstein (1989).

The basic idea of the proof is that in states $(0,0)$ and $(1,0)$ the obvious equilibrium is (A, A) - given $p < \frac{1}{2}$. Using this as the start of an induction, one has that up to the observation of $T-1$ for each player it is optimal to play A. The consistent belief $z = \frac{\varepsilon}{\varepsilon+\varepsilon(1-\varepsilon)}$ to be at the first of two indistinguishable "outcomes" $(T, T-1)$ and (T, T) for player 1 [or $(T-1, T-1)$ and $(T, T-1)$ for player 2] is greater than $\frac{1}{2}$. Given the stated belief and that up to state $(T-1, T-1)$ [or $(T, T-1)$ for player 2] the best reply of the other player is A, it is a best a answer to choose A if the information set is reached because this decision is independent of the strategy of the other player at the second indistinguishable "outcome" in the information set. By induction this is true for every observed T.

are "divided" by the corresponding information sets of the other player in a way that the first part (the states that are in the corresponding first information set of the other player) is smaller than the rest of the information set. Therefore the next informational structure that would yield the Rubinstein result is where both player cannot distinguish three succeeding numbers and the information sets overlap exactly the way that in each set three "outcomes" are in each of the corresponding sets of the other player.

[5] Piccione and Rubinstein (1996) and Aumann et al. (1996) in addition to a special issue of *Games and Economic Behavior* 1996 (forthcoming) cover the problem of imperfect recall in an example of an absent-minded driver. An application to the centipede game can be found in Dulleck and Oechssler (1996).

The following proposition states the main results of the paper for the extended game where one player suffers from non-distinguishability.

Proposition 2 *If L is not too large relative to M then there exists a Nash equilibrium such that both players play B if their machine sent $t \geq T$ messages and A in all other cases, given one player suffers from non-distinguishability such that he cannot distinguish among the $t \in \{T, T+1, ..., T+l\}$.*

Proof. First we proof the result given player 1 suffers from non-distinguishability such that he cannot distinguish among the $t \in \{T, T+1, ..., T+l\}$. To show that this is an equilibrium, we proceed as in Rubinstein (1989) up to "outcome" $(T, T-1)$. See the argument above or Rubinstein (1989) for the formal proof. For any state of the world where $t < T$ it is always a best reply to play A regardless of the state of the world.

If Player 1 cannot distinguish "outcome" $(T, T-1)$ from its $2l+1$ successors he forms the belief $\widetilde{z} = \frac{\varepsilon}{\sum_{i=0}^{2l+1} \varepsilon(1-\varepsilon)^i}$ to be at "outcome" $(T, T-1)$ where player 2 plays A for sure in the specified equilibrium. If $\frac{1-\widetilde{z}}{\widetilde{z}} M = \sum_{i=1}^{2l+1} (1-\varepsilon)^i > L$ then playing B is optimal at this information set given the specified strategy of player 2 (to play B if he observes a $t \geq T$). Given player 1´s strategy the best reply by player 2 is to play B whenever he observes $t \in \{T, T+1, ..., T+l\}$.

Given that players play B whenever they observe a $t \in \{T, T+1, ..., T+l\}$, induction implies that they do so for $t > T+l$. At "outcome" $(T+l+1, T+l)$ player 1´s best reply in the information set where he observes that any $t \geq T+l+1$ messages are sent is to play B given the strategy of the other player who plays B at the two indistinguishable "outcomes" in this information set.

If player 2 cannot distinguish between "outcomes" where he observes $t \in \{T, T+1, ..., T+l\}$ the belief to be at "outcome" (T, T) instead of one of the non-distinguishable successors is again $\widetilde{z}$. The same argument applies as in the case where player 1 suffers from non-distinguishability. ■

Next I analyse the case that one player suffers only with a probability of η from non-distinguishability at the periods where he observe any $t \in \{T, T+1, ..., T+l\}$. Without loss of generality I assume that player one may suffer from non-distinguishability. Let $\widetilde{z} = \frac{\varepsilon}{\sum_{i=0}^{2l+1} \varepsilon(1-\varepsilon)^i}$.

Proposition 3 *If* $\frac{1-\tilde{z}}{\tilde{z}}M > L$ *and* $(1-\eta) \leq \frac{(2-\varepsilon)M}{M+L}$ *then there exists a Nash equilibrium where both players play* B *whenever they observe a* $t \geq T+1$ *given player* 1 *suffers with probability* η *from non-distinguishability such that he cannot distinguish among the* $t \in \{T, T+1, ..., T+l\}$*. For* $M = L$ *this implies that such an equilibrium exists if* $\eta \geq \frac{1}{2}\varepsilon$*.*

The following strategies are the equilibrium in question: If Player 1 suffers from non-distinguishability he chooses B if he observes a $t \geq T$ and A otherwise. If player 1 does not suffer from non-distinguishability, he chooses B whenever he observes a $t > T$ and A otherwise. Player 2 chooses B if he observes a $t \geq T$ and A otherwise. I will proof that this is an equilibrium given the stated restrictions on M, L and η.

Proof. I proof that the strategies are best reply strategies. Up to "outcome" $(T-1, T-1)$ the strategies follow from the argument in the proof of proposition 1.

If player 1 suffers from non-distinguishability proposition 2 ensures that his best reply is given as described given the stated strategy of player 2. If he does not suffer from non-distinguishability his best reply up to where he observes T is to play A by the argument in the proof of proposition 1. If he observes a $t > T$ then given the stated strategy of player 2 his best reply is to play B because player 2 plays B at both of the "outcomes" in his information set. Therefore given the strategy of player 2 player 1´s strategies are best answers.

Given the strategy of player 1 the payoff to player 2 if he plays B whenever he observes that exactly T messages have been sent is given by

$$\eta M + (1-\eta)(z(-L) + (1-z)M) \tag{1}$$

where $z = \frac{\varepsilon}{\varepsilon + \varepsilon(1-\varepsilon)}$ is the consistent belief that the state is (T, T) instead of $(T+1, T)$. Player 2´s payoff is 0 if he chooses A. Therefore B is a best reply if (1)≥ 0. This is equivalent to $(1-\eta) \leq \frac{(2-\varepsilon)M}{M+L}$ as stated in the proposition.

Given the observation of any $t > T$ by player 2 his best reply is to choose B because player 1 chooses B at both "outcomes" in the information set in question.

For $M = L$ the condition $(1-\eta) \leq \frac{(2-\varepsilon)M}{M+L}$ simplifies to $\eta \geq \frac{1}{2}\varepsilon$. ∎

Therefore given the probability to suffer from non-distinguishability is large enough compared to the probability that a message gets lost there exists a Nash equilibrium where players condition on the number of messages sent.

Note, given $M = L$ this probability may be infinitesimal small given a small ε.

When the (potentially) non-distinguishable states are reached it is optimal to play B. In contrast to Rubinstein´s (1989) result for the case that the number of messages sent is limited (which is equivalent to $l = \infty$) it is sufficient that non-distinguishabilty appears only in earlier stages ($l < \infty$). Once it is optimal to play B at any stage then induction leads to the result that in the succeeding stages, playing B is the optimal strategy.

In the case where one player suffers from non-distinguishability, players in the payoff dominant equilibrium use a strategy where they condition their action on the number of messages sent.

Corollary 1 *If one player suffers from non-distinguishability the optimal strategies for players who observe that exactly T messages have been sent differ compared to the case without non-distinguishability, given the observed number of messages sent is greater then the number where the non- distinguishability affects the utilizable information.*

Therefore, the decision of players does not only depend on the information they have at the point of time where they have to take their decision. It also depends on the information available to them at an earlier point in time. Even though the number players observe in Rubinstein´s original game and the presented extended version is the same, the best-reply-strategies differ because of an informational deficiency which could have arisen at an earlier stage in the game (but which actually might not have had any effect on the utilizable information).

Corollary 2 *The expected payoffs in the coordination game that is the basis for the E-mail game vary non-monotonically in the information structure.*

The expected payoff under common knowledge in the E-mail game is $\Pi^e = M$. Given Rubinstein´s "almost common knowledge" the expected payoff is $\Pi^e = pM$. Introducing non-distinguishability and therefore a further reduction of utilizable information at only one-point in time one gets $\lim_{\varepsilon \longrightarrow 0} \Pi^e = M$. In the case that only one player is informed of the state of the world and no communication takes place, one is back at $\Pi^e = pM$.

4 Conclusions

Rubinstein (1989) employs the Electronic Mail game to illustrate that the payoffs vary discontinuously in the assumed information structure, i.e. "almost common knowledge" leads to different optimal behavior compared to the optimal behavior under common knowledge. In this game bounded rationality leads to a continuity in the expected payoffs if one reduces the quality of the available information (from common knowledge to "almost common knowledge"). Having at one stage in time a different optimal strategy, induction is carried out in a different way and leads to different optimal behavior.

Rubinstein´s case shows that knowing less implies a welfare decrease for both agents. The extension of his game illustrates that knowing even less than in the original game with imperfect information increases welfare almost to the level that is reached under common knowledge. After all, given the worst situation in this game (nobody or only player 1 knows the state of the world) we have again a unique equilibrium as under "almost common knowledge" with the low expected payoffs. Therefore this is an example of non-monotonicity in available information.

Another paradoxical aspect of this game is that even if agents know the difference between 16, 17 and 18 and they observe 17, they play B if they cannot distinguish between let us say 7 and 8. A local deficiency in information processing abilities changes the optimal strategy even though at the decision making point in time the available information is the same as in the case where no local deficiency exists.

References

[1] Aumann, R.J., Hart, S. and Perry, M. (1996), "The Absent-Minded Driver", Discussion Paper #94, Hebrew University of Jerusalem (*Games and Economic Behavior*, forthcoming)

[2] Dulleck, Uwe and Oechssler, Jörg (1996), "The Absent-Minded Centipede", *Economics Letters*, forthcoming

[3] Fagin, Ronald, Halpern, Joseph Y., Moses, Yoram, Vardi, Moshe Y. (1995), "*Reasoning About Knowledge*", MIT Press, Cambridge

[4] McKelvey, R. and Palfrey, T. (1992), "An experimental study of the centipede game", *Econometrica*, 60, p. 803-836

[5] Osborne, Martin J. and Rubinstein, Ariel (1994), "*A course in game theory*", MIT Press, Cambridge

[6] Piccione, M. and Rubinstein, Ariel (1996), "On the Interpretation of Decision Problems with Imperfect Recall", *Games and Economic Behavior*, forthcoming

[7] Rosenthal, R. (1981), "Games of perfect information, predatory pricing and chain-store paradox", *Journal of Economic Theory*, 25, p. 92-100

[8] Rubinstein, Ariel (1989), "The Electronic Mail Game: Strategic Behavior Under "Almost Common Knowledge" ", *AER*, Vol. 79, No. 3, p. 385-391