Third– and Lower–Price Auctions*

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Abstract

This paper solves the equilibrium bid functions of third– and lower–price auctions. In these auctions, equilibrium bids *exceed* bidders' valuations, and bidders raise their bids if one moves to a lower price auction, and lower bids if the number of bidders is increased. Third– and lower–price auctions are unappealing under risk aversion, which in turn may make them appealing when the auction is a substitute for small scale gambling, as in many Internet auctions. Moreover, in the presence of a corrupt agent-auctioneer, an auction may turn out to be third– or lower–price, even though it was set up as a second–price or hybrid English auction.

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1 Introduction

In the analysis of independent private value auctions it is a common exercise to compare equilibrium bidding in first– and second–price auctions. While truthful bidding is a (weakly) dominant strategy in second–price auctions, bidders necessarily "shade" their bids below their valuations in a first–price auction. Therefore, switching from a second– to a first–price auction leads to lower bidding. However, if the number of bidders is increased, the participants of a first–price auction raise their bids which in turn reduces the gap between the two equilibrium bidding rules.

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From first– and second–price auctions one may extrapolate and design third– and lower–price auctions, where the winner pays the third–highest bid, or more generally the k–th highest bid, for $k > 2$. This raises the question: can one also extrapolate how equilibrium bidding changes if one moves from second– to third– and lower–price auctions and if the number of bidders is increased?

The present paper solves equilibrium bid functions of third– and lower–price auctions for a large class of distribution functions of bidders' valuations, assuming the symmetric independent private values framework and risk neutrality. The solutions imply the following general properties of the k–price auction, for $k > 2$,

1. equilibrium bids *exceed* bidders' valuations (the opposite of "shading")

2. moving to a lower–price auction leads to higher bidding (equilibrium bids increase in k)

3. equilibrium bids diminish if the number of bidders is increased

4. moving to a lower–price auction tends to increase the variance of the equilibrium price.

These properties delineate a general pattern of how equilibrium bidding rules change as we move from first– to second–, and lower–price auction all the way to the n–th-price auction, where the winner pays "only" the lowest bid, and they may contribute to explain the predominance of first–price auctions.

Third–price auctions were considered for the first time by Kagel and Levin (1993), who solved the equilibrium bid function assuming uniformly distributed valuations and who identified some further general properties, but did not give a general solution of equilibrium bid functions.

Third–price auctions have been useful to test the predictive power of auction theory in laboratory experiments (see Kagel and Levin (1993)). They may also be appealing when the auction involves some small scale gambling aspect, as in many Internet auctions. Moreover, in the presence of corruption, bidders may bid as if the auction were a third- or lower–price auction, even if the auction has been set up as second-price or hybrid English auction.[1]

[1]A hypbrid English auction is an English auction in which the auctioneer agrees to accept and execute sealed–bids. Most English auctions are such hybrid auctions.

2 Assumptions

Consider a k–price auction where a single unit is sold to $n \geq k$ risk neutral bidders. This auction is characterized by three rules: 1) the item is awarded to the highest bidder (ties are handled by some allocation rule); 2) the winner pays the k–th highest bid, $k \in \{n, n-1, n-2, \ldots, 1\}$; 3) only the winner pays. The familiar first– and second–price auctions are obtained for $k = n$ and $k = n - 1$, and third– and lower–price auctions are obtained by setting $k = n - 2$ and $k < n - 2$.

The analysis assumes the symmetric independent private values framework. Each bidder's valuation is private information, and all bidders view their rival's valuations as *iid* random variables, drawn from the continuous distribution function (c.d.f.) $F(v)$, with density (p.d.f.) $f(v) > 0$, and support $[0, \bar{v}]$, $\bar{v} > 0$. The p.d.f. satisfies the usual regularity condition and is log–concave.[2] All bidders are risk neutral.

Throughout the paper, let $V_n, V_{n-1}, \ldots, V_1$ denote the largest, second largest to lowest valuation of the entire sample of n bidders. Similarly, denote the largest to lowest valuation of the smaller sample of $n-1$ valuations by $Y_{n-1}, Y_{n-2}, \ldots, Y_1$. The two sets of order statistics of valuations differ exclusively in the underlying sample size.

The c.d.f. of the order statistic Y_{n-1} is $F_{Y_{n-1}}(x) = F(x)^{n-1}$. And the conditional p.d.f. of Y_{n-k+1} for $k > n-1$, conditional on $Y_{n-1} < x$, is[3]

$$f_{Y_{n-k+1}|Y_{n-1}<x}(y) = \frac{(n-1)!}{(n-k)!(k-2)!} \frac{f(y)F(y)^{n-k}\left(F(x) - F(y)\right)^{k-2}}{F(x)^{n-1}}.$$

Bidders' strategy is their bid function $b_k(v) : [0, \bar{v}] \to \mathbb{R}_+$ and participation rule $\xi_k(v) : [0, \bar{v}] \to \{0, 1\}$. We characterize symmetric equilibria where each bidder bids according to the same bid function $b_k^*(v)$, and we assume that the minimum bid is equal to 0 so that bidders participate at all valuations.

3 Third–Price Auctions

To solve the third–price auction, proceed as follows. Assume the equilibrium is symmetric and the equilibrium strategy $b_3^* : [0, 1] \to [0, 1]$ is strict monotone increasing (this will be confirmed later on). Consider one bidder and suppose all other bidders play the strategy b_3^*. Then

[2] F is log–concave if $\ln(F)$ is concave. Log concavity is frequently assumed in information economics. It is assured by all standard distribution functions; see Bagnoli and Bergstrom (1989).

[3] For reviews of order statistics see David (1970) or Arnold, Balakrishnan and Nagaraja (1992).

that bidder need only consider bids from the range $[b_3^*(0), b_3^*(1)]$ (bidding outside this range is either unnecessarily high or too low). And all relevant deviating bids can be generated by inserting $x \in [0,1]$ into the equilibrium strategy, $b_3^*(x)$, i.e. by bidding as if the valuation were $x \in [0,1]$ rather than the true valuation v.

Let $U(x,v)$ denote bidders' payoff and $\rho(x)$ their probability of winning the auction provided 1) the true valuation is v, 2) all others play the strategy b_3^*, 3) the bidder bids $b_3^*(x)$.

Using the notation of order statistics and the assumed monotonicity of b_3^* one obtains

$$U(x,v) := \rho(x)\left(v - E[b_3^*(Y_{n-2}) \mid b_3^*(Y_{n-1}) < b_3^*(x)]\right) \tag{1}$$

$$\rho(x) := \Pr\{b_3^*(Y_{n-1}) < b_3^*(x)\} = F(x)^{n-1}. \tag{2}$$

Combine these, compute the conditional expected value, and one gets

$$\begin{aligned} U(x,v) =& \rho(x)\left(v - \int_0^x b_3^*(y) f_{Y_{n-2}|Y_{n-1}<x}(y)dy\right) \\ =& vF(x)^{n-1} - (n-1)\int_0^x b_3^*(y)\left(F(x) - F(y)\right) dF(y)^{n-2}. \end{aligned} \tag{3}$$

The strategy b_3^* is a symmetric Bayesian Nash equilibrium iff it does not pay to deviate from the equilibrium bid $b_3^*(v)$

$$v \in \arg\max_x U(x,v), \quad \forall v. \tag{4}$$

We will use this requirement together with (3) to find the equilibrium.

Proposition 1 *Consider the third-price auction, where the highest bidder wins and pays "only" the third highest bid. Then, the equilibrium bid function is*

$$b_3^*(v) = v + \frac{F(v)}{(n-2)f(v)} \quad \textit{(3-rd price auction)}. \tag{5}$$

Proof The necessary condition for (4) is the following identity in v

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} U(x,v)|_{x=v} \\ &= v(n-1)F(v)^{n-2}f(v) - (n-1)f(v)\int_0^v b_3^*(y)dF(y)^{n-2}. \end{aligned}$$

Divide this identity by $(n-1)f(v)$, then differentiate with respect to v, and one obtains

$$b_3^*(v)\frac{d}{dv}F(v)^{n-2} = F(v)^{n-2} + v\frac{d}{dv}F(v)^{n-2}.$$

Straightforward rearranging gives the asserted equilibrium strategy (5).

The necessary condition is also sufficient if 1) the derived strategy b_3^* is strict monotone increasing (which was assumed when we wrote the necessary condition) and 2) if each stationary point is indeed a global maximum. The first requirement follows immediately from the assumed log–concavity of F (note: $F(v)/f(v)$ is increasing). And the second requirement follows from the fact that $U(x,v)$ is increasing in x iff $x < v$ and decreasing iff $x > v$ ("pseudoconcavity" of $U(x,v)$ in x). □

4 Generalization

We now generalize and compute the equilibrium bid functions of second-, third-, fourth-, and lower–price auctions.

PROPOSITION 2 *Consider the k-price auction where the highest bidder wins and pays "only" the k-th highest bid, $k \in \{2,3,\dots,n\}$. Then, the equilibrium bid function is*

$$b_k^*(v) = v + \frac{k-2}{n-k+1}\,\frac{F(v)}{f(v)} \quad \text{(k-price auction).} \tag{6}$$

PROOF By a well–known result, the symmetric equilibria of all auction games that award the item to the highest bidder give rise to the same expected payments.[4] This applies in particular to the comparison of third– and lower–price auctions with the second–price auction. As is well–known, truthful bidding is the unique symmetric equilibrium of second–price auction, $b_2^*(v)$. Therefore, in a second–price auction the price is made by Y_{n-1}, whereas in a k–price auction it is made by the random variable $b_k^*(Y_{n-k+1})$. Hence, the equality of expected payments requires, for all v,

$$E[Y_{n-1} \mid Y_{n-1} < v] = E[b_k^*(Y_{n-k+1}) \mid Y_{n-1} < v],$$

or equivalently

$$\int_0^v x(n-1)F(x)^{n-2}f(x)dx = \frac{(n-1)!}{(n-k)!(k-2)!}\int_0^v b_k^*(x)f(x)F(x)^{n-k}\left(F(v)-F(x)\right)^{k-2}dx. \tag{7}$$

Differentiate the latter identity $(k-1)$–times, dividing by $f(v)$ after each round of differentiation (including the last). Then, one obtains for

[4] See, Riley and Samuelson (1981) or Myerson (1981).

the right–hand side (RHS) of (7)

$$\text{RHS} = b_k^*(v)\frac{(n-1)!}{(n-k)!}F(v)^{n-k}.$$

To apply the same procedure to the left–hand side (LHS) of (7) is more tedious. Use the product rule of differentiation,

$$g(x)h(x))^{(n)} = \sum_0^n \binom{n}{r} g^{(r)}(x)h^{(n-r)}(x),$$

and the rule for the r–th derivative of the polynomial $\psi(x) := x^n$, which is $\psi^{(r)}(x) = r!\binom{n}{r}x^{n-r}$, and one obtains after some rearranging

$$\text{LHS} = \frac{(n-1)!}{(n-k)!}F(v)^{n-k}\left(v + \frac{k-2}{n-k+1}\frac{F(v)}{f(v)}\right).$$

Equate RHS and LHS, and one has the asserted bid function. The assumed monotonicity is again assured by the log–concavity of F. □

REMARK 1 *If valuations are uniformly distributed on the support* $[0,1]$*, one has* $b_k^*(v) = \frac{n-1}{n-k+1}v$*, as in Kagel and Levin (1993).*

REMARK 2 *In the case of a uniform distribution of valuations it is easy to see that the variance of the random equilibrium price* P_k

$$Var(P_k) = Var\left(b_k^*(V_{(n-k+1)})\right) = \frac{k(n-1)}{(n-k+1)(n+1)^2(n+2)^2}.$$

is strict monotone increasing in k*. This suggests that second–, third, and lower–price auctions are appealing only to risk lovers.*[5]

Third– and lower–price auctions have four striking properties: 1) bids are higher than the own valuation; 2) equilibrium bids increase if one moves to a lower–price auction; 3) equilibrium bids diminish as the number of bidders is increased; and 4), the risk of the random equilibrium price tends to increase as one moves to lower–price auctions.

Once one has figured out why it pays to "speculate" and bid higher than one's own valuation, it is easy to interpret the third property. Just keep in mind that a rational bidder may get "burned" and suffer a loss because the k–th highest bid is above the own valuation. As the number of bidders is increased, it becomes more likely that the k–th highest bid is in close vicinity to one's own valuation. Therefore, it makes sense to bid more conservatively if the number of bidders is increased.

[5]For an assessment of risk attributes in terms of second–order stochastic dominance instead of variance see Wolfstetter (1999a, p. 155).

5 Third- and Lower-Price Auctions and Gambling

The fourth property suggests that a risk averse seller who faces risk neutral bidders should always prefer lower order k–price auctions and should most prefer the first–price auction. While the second–price auction is always appealing because of its overwhelming strategic simplicity, third– and lower–price auctions are strategically just as complicated as the first–price auction, but in addition expose the seller to unnecessary risk.

However, risk neutrality or risk aversion is often not the appropriate framework. Many auctions have a gambling aspect. Internet auctions are a case in point.

If bidders or the auctioneer have a local risk preference (which can be captured by a Friedman–Savage utility function), third- and lower-price auctions may be the right auction format to satisfy a demand for gambling (as suggested by Monderer and Tannenholz (1999)).

The gambling aspect of third- and lower-price auctions is potentially interesting in countries where explicit gambling is illegal or tightly regulated, and when small amounts of money are involved, as in many "c2c" Internet auctions. On the other hand, governments are usually quick to detect any infringement of their monopoly right for issuing gambling licenses, and it is hard to imagine that they would tolerate an auction format that serve as a substitute for gambling.

6 Third- and Lower-Price Auctions and Corruption

Even though a risk averse seller who deals with risk neutral bidders should not intentionally set up a third– or lower–price auction, collusion may effectively transform a second–price or hybrid English auction into such an auction. In this indirect manner third– and lower–price auctions gain practical significance, even if the auction was set up as a second–price or hybrid English auction.

Collusion may take one of two forms: *corruption* and *auction rings*. Of course, corruption and auction rings may occur either alone or in combination.

If the auctioneer is an agent of the seller, some bidder(s) may bribe the auctioneer to withhold some bids. Therefore, corruption is an issue.

Typically, the seller responds to the fear of corruption by imposing certain publication requirements, adopting the age–old principle of "legality through publicity". For example, U. S. Federal Law mandates the publication of the winning bid in all government procurements. The rationale for this requirement is precisely to make corruption more difficult.

If the highest bid is published, the auctioneer cannot be bribed to withhold the highest bid. Because the highest bidder would find out and complain. But the requirement to publish the highest bid cannot prevent the suppression of the second highest and lower bids. This is where corruption ties in with third– and lower–price auctions.

Based on these observations we now develop a simple model of corruption in second-price auctions, derive some results, and then close with some remarks on alternative models of corruption.

Suppose the auctioneer has been corrupted and accepts an agreement with some bidder(s) to suppress the second highest to k–th highest bids, where k is some positive integer between $n-2$ and 1. Generally, the suppression of all but the winning bid arouses suspicion; therefore, k should be larger than 1.

Since the highest bid cannot be suppressed, and since the arrangement is made before bids have come in, the bidder who bribes the auctioneer cannot be sure to actually win the auction and benefit from this arrangement. The bribe pays off only if that bidder has the highest bid, and if the highest bid among all bids that are not withdrawn by the auctioneer happen to be smaller than that bidder's valuation.

Specifically, we assume that one bidder has made an agreement with the auctioneer to suppress the second to $k+1$ highest bids. Since the auction is second-price, it follows that the price this bidder pays in the event of winning is equal to the k-th highest bid. Other bidders are not aware of this agreement, and they bid truthfully which is their (weakly) dominant strategy. Therefore, the bidder wins with a bid equal to b if and only if $b > V_{(n-1)}$, in which case he pays a price equal to $V_{(n-k)}$ or equivalently a $Y_{(n-k+1)}$.

It follows that the bribing bidder's optimal bid solves the following maximization problem

$$\max_b F(b)^{n-1}\left(v - \int_0^b y f_{Y_{n-k+1}|Y_{n-1}<b}(y)dy\right). \tag{8}$$

The first–order conditions are

$$F(b)^{n-2}v = \gamma \int_0^b y f(y) F(y)^{n-k}\left(F(b) - F(y)\right)^{k-3} dy, \ \forall v \tag{9}$$

$$\gamma := \frac{(n-2)!}{(n-k)!(k-2)!}. \tag{10}$$

Of course, it is optimal to bid higher than the true valuation. Indeed, any bid $\bar{b} < v$ is inferior to $b = v$, because raising the bid to $b = v$ makes winning more likely, given that all others bid truthfully, and winning is beneficial at $b = v$, because the price to be paid is independent and below the own bid.

Since others bid truthfully, one may guess that the optimal bid is below the equilibrium bid $b_k^*(v)$ that applies when corruption is common knowledge, because all rivals bid less aggressively if corruption is kept secret. However, this conjecture does not confirm. Indeed, it may be optimal to bid as if the auction were a k–price auction. Indeed, this surprising property holds precisely if the c.d.f. of valuations is uniform.

PROPOSITION 3 *Consider a second-price auction, and suppose one bidder has a secret agreement with the auctioneer to withhold the second to (k+1) highest bids* $k \in \{3, 4, \ldots, n\}$. *If valuations are uniformly distributed, that bidder should bid as if the auction were a k–price auction*

$$b_k^*(v) = \frac{n-1}{n-k+1} v. \tag{11}$$

PROOF The proof is in two steps: 1) we show that (11) is the one and only solution in linear strategies, and 2) we prove uniqueness.

1) Suppose $v = \alpha b$, $\alpha > 0$ (linearity) and $F(v) = v$ (uniform distribution). Then, the first–order conditions (9) can be rewritten in the following form

$$b^{n-1}\alpha = \gamma \int_0^b y^{n-k+1}(b-y)^{k-3} dy, \quad \forall b. \tag{12}$$

Differentiate this identity (k-3)–times with respect to b, and one obtains

$$(n-1)\cdot(n-2)\cdots(n-k+3) b^{n-k+2}\alpha = \frac{\gamma(k-3)!}{n-k+2} b^{n-k+2}, \ \forall b.$$

Rearrange, and one has $\alpha = \frac{n-k+1}{n-1}$, and hence $b = \frac{n-1}{n-k+1} v$.

2) Consider b as given. The left–hand–side of (9) is increasing in v, whereas the right–hand–side is independent of v. Therefore, the two sides are equal at most at one v. Hence the solution (11) is unique. □

There are several alternative models of corruption in auctions. For example other bidders may suspect or even know that the agent-auctioneer is corrupt, and then also adjust their bidding. Moreover, the agent-auctioneer may be part of more complex collusion schemes, where bidders may be induced to withdraw or even change their bid. These and related issues are further examined in Lengwiler and Wolfstetter (2000).

7 Conclusions

This paper has developed a procedure to solve the equilibrium strategies of third- and lower-price auctions in the symmetric, independent, private values framework. In addition, we characterized the risk attributes of these auction formats, which suggests that they are appealing to (local) risk lovers, as a substitute for gambling. Moreover, we showed that, in the presence of a corrupt agent-auctioneer, bidders may find it in their interest to bid as if they participated in a third- or lower-price auction, even though the auction has been set up as a second-price or hybrid English auction.

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