Oligopoly: Cournot/Bertrand/Stackelberg

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Common Structure

**Common Assumptions**

1. Risk neutral firms maximize profits.
2. Firms produce a homogeneous product.
3. All cost functions are known to all firms.
4. Firms meet once and cannot make binding agreements.

Each firm takes into account that their profits depend on actions of the other firms (strategic interaction).
Alternative Market Models

Cournot (1839): firms simultaneously choose quantities

1. Firms simultaneously select quantities (capacities):
   \[ q_i \in [0, D(0)], \ i \in I. \]

2. Price is chosen (by an auctioneer) to clear the market
   \[ D(p) = \sum_I q_i. \]

Bertrand (1883): firms simultaneously select prices

1. Firms simultaneously select prices: \( p_i \in [0, p^m], \ i \in I. \)

2. The cheapest supplier \( l \) gets the whole demand \( q_l = D(p_l). \)
   (The lowest bid gets the contract). There are no capacity constraints. Each firm can serve the whole market.
Alternative Market Models

Stackelberg (1934): firms sequentially select quantities

1. The leader $l$ selects his quantity (capacity): $q_l \in [0, D(0)]$.
2. The followers observe $q_l$ and select their quantities: $q_i \in [0, D(0)]$.
3. Price is chosen (by an auctioneer) to clear the market $D(p) = \sum_I q_i$. 

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Oligopoly: Cournot/Bertrand/Stackelberg
Cournot Game: the interaction

Describing strategic interaction as a game in normal form:

**Players:** Firms $A$ and $B$

*(pure) strategies, strategy-space :*

$$\{q_A, q_B\} \in S = [0, \bar{q}] \times [0, \bar{q}]; \quad \bar{q} = D(0)$$

**Pay-off functions:**

$$\pi_A(q_A, q_B) = q_A \cdot P(q_A + q_B) - K_A(q_A)$$

$$\pi_B(q_A, q_B) = q_B \cdot P(q_A + q_B) - K_B(q_B)$$
Cournot Game: the solution

Solution: Cournot–Nash–equilibrium:

A combination of strategies \((q_A^*, q_B^*) \in S\) so that both firms select an optimal strategy given the choice of the other firm:

\[
q_A^* = \arg \max_{q_A} \pi_A(q_A, q_B^*), \quad q_B^* = \arg \max_{q_B} \pi_B(q_A^*, q_B)
\]
Finding the Solution

In principle, the solution can be obtained from the Kuhn-Tucker conditions. Let \( q = q_A + q_B \) then \( (q^*_A, q^*_B) \) solve

\[
\frac{\partial \pi_A(q_A, q_B)}{\partial q_A} = [P'(q)q_A + P(q) - K'_A] \leq 0, \quad [\cdot]q_A = 0
\]

\[
\frac{\partial \pi_B(q_A, q_B)}{\partial q_B} \leq 0, \quad [\cdot]q_B = 0
\]
Reaction Functions

Useful tools: ‘reaction functions’ and ’inclusive reaction functions’:

\[ Q_A(q_B) := \arg \max_{q_A} \pi_A(q_A, q_B) =: Q_A^i(q) \]

\[ Q_B(q_A) := \arg \max_{q_B} \pi_B(q_A, q_B) =: Q_B^i(q) \]

The correspondence \( \psi \)

\[ \psi(q_A, q_B) = \{ Q_A(q_B), Q_B(q_A) \} \]

is a mapping from \( S \) into itself. \( (q_A^*, q_B^*) \) is a fixed–point of this mapping.
Slope of reaction functions

From the first order condition

\[
\frac{dQ_A}{dq_B} = - \frac{\partial^2 \pi_A}{\partial q_A \partial q_B} \left( \frac{\partial^2 \pi_A}{(\partial q_A)^2} \right)
\]

By the second order condition:

\[
\text{sign } \frac{dQ_A}{dq_B} = \text{sign } \frac{\partial^2 \pi_A}{\partial q_A \partial q_B} = \text{sign } P'' q_A + P'
\]

Assumed to be negative (recall \( P' < 0 \)).
Strategic Substitutes

**strategic substitutes**: the firm responds to an increase of the rival’s quantity with a reduction of its own quantity.

**in general**: If an increase of another players’s choice variable leads to a decrease of the player’s optimal level of his choice variable.
Cournot Equilibrium
Existence: mixed strategies

Existence in mixed strategies

Since $S$ is compact (closed and bounded) and payoffs are continuous functions of strategies the existence of an equilibrium (possibly in mixed strategies) is assured (Glicksberg (1952)).

If $S$ would be replaced by a discrete grid of quantities existence would be assured by Nash (1952).
Existence: pure strategies

Existence in pure strategies

Debreu (1952): there exists a pure-strategy equilibrium if

1. $S$ is convex and compact,
2. all payoff functions are continuous in the strategy profiles $s \in S$,
3. all payoff functions are quasiconcave in the players’ own strategies.
Uniqueness in pure strategies

Kohlstad & Mathiesen (1987) provide necessary and sufficient conditions for uniqueness in pure strategies. Essentially, a small change of the firm’s output must have a larger impact on the firms own profit than a similar change of output on part of the rival firms.
Uniqueness

Assume that for \( j = A, B \):

\[
\frac{dQ_j}{dq} = - \frac{P''(q)Q_j(q) + P'(q)}{P'(q) - K_j''(Q_j(q))} < 0.
\]

Define the aggregate inclusive reaction function:

\[ Q^i(q) = Q^i_A(q) + Q^i_B(q). \]

1. \( Q^i \) is a mapping from \([0, q^M]\) into itself. Every fixed point \( q_o \) of this mapping generates a Nash equilibrium \((Q^i_A(q_o), Q^i_B(q_o))\) and vice versa.

2. \( Q^i_j(0) > 0, \ Q^i_j(q^M) = 0, \ j = A, B, \)
   hence \( Q^i(0) > 0, \ Q^i(q^M) = 0 \)

3. \( Q^i \) is continuously differentiable, with \( dQ^i/dq < 0 \).

The Cournot game has a unique Nash equilibrium, because \( Q^i \) cuts the 45 degree line exactly once.
Iterated Elimination of Dominated Strategies

Example:
Linear demand $P(q) = a - q$, constant marginal cost $c$.

With two players:
The set of rationalizable strategies is equal to the Nash-equilibrium strategies $(q_A^c, q_B^c)$.

Iterated elimination of dominated strategies allows you to lower the upper bound and raise the lower bound of rationalizable strategies until you approach the Nash-equilibrium.
Iterated Elimination of Dominated Strategies

With three players:

The set of rationalizable strategies is equal to $[0, q_M]^3$. While no player will ever set $q_i > q^M = Q_i(0)$, $Q_i(2q^M) = 0$ so we cannot increase the lower bound. Hence, iterated elimination of dominated strategies is of little use to predict behavior.

For a detailed presentation see Gibbons 1992, Game Theory for Applied Economists, Princeton University Press. pp.18ff
Mark-Up

We look at general case \( n \geq 2 \) firms \( (q = \sum_{i=1}^{n} q_i) \).

**FOC**: 
\[
P(q) - K_i'(q_i) + q_i P'(q) = 0
\]

Industry cost is not minimized (marginal cost are different for different firms).

Rewrite (FOC) as mark-up rule (Lerner index):
\[
\frac{P - K_i'}{P} = \frac{q_i}{q} \frac{1}{-\epsilon(P)}
\]

Mark-up is proportional to market share and inversely proportional to elasticity of demand.
Symmetric Case

In the symmetric case we obtain:

\[
\frac{P - K'}{P} = \frac{1}{n} \cdot \frac{1}{-\epsilon(P)}
\]

As the number of firms \(n\) becomes larger, we move from monopoly to perfect competition.
Oligopoly is somewhere between perfect competition and monopoly.

\[ q^m < q^* < q^k; \]
\[ p^m > P(q^*) > p^k; \]
\[ \pi^m > \sum_{i=1}^{n} \pi_i(q^*) > \pi^k \]

The larger the number of firms the closer we approach perfect competition.

Results make sense, but price determination is implausible. In most markets firms determine prices themselves.
Horizontal Merger and Market Power

Suppose we have $n$ identical firms in the market. Marginal cost is $c$ and inverse demand is given by: $P(q) = a - bq$.

$$q^*(n) = \frac{a - c}{b(1 + n)}$$

and

$$\pi(n) = (P(nq^*(n)) - c)q^*(n) = \frac{(a - c)^2}{b(1 + n)^2}$$

Two firms are contemplating a merger. Will it increase market power? Will it increase profits?
Mergers don’t pay off

**Proposition:** For \( n > 2 \), a horizontal merger of two firms increases the profits of the industry but is not profitable for the merging firms.

**Proof:**

1. Obviously it increases profits of the other firms because:
   \[ \pi(n - 1) > \pi(n). \]

2. Industry profits increase:
   \[ n^2 - n - 1 > 0 \implies (n - 1)\pi(n - 1) > n\pi(n) \]

3. Merging firms profit decrease:
   \[ 1 + 2n - n^2 < 0 \implies \pi(n - 1) < 2\pi(n) \]
Bertrand (1883): firms select prices

Assumptions:

1. Risk neutral firms maximize profits.
2. Firms produce a homogeneous product.
3. All cost functions are known to all firms.
4. Firms meet once and cannot make binding agreements.
5. Firms simultaneously select prices: \( p_i \in [0, p^m], \ i \in I \).
6. The cheapest supplier \( l \) gets the whole demand \( q_l = D(p_l) \).

(The lowest bid gets the contract). There are no capacity constraints. Each firm can serve the whole market.
The Bertrand Game:

Game in normal form:

**Players:** Firms $A$ and $B$

*(pure) strategies, strategy-space:*

$$p_A \in [0, p^m]; \quad p_B \in [0, p^m]$$

**Pay-off function ($A$):**

$$\pi_A(p_A, p_B) = \begin{cases} 0 & \text{if } p_A > p_B \\ \frac{1}{2} D(p_A)(p_A - c) & \text{if } p_A = p_B \\ D(p_A)(p_A - c) & \text{if } p_A < p_B \end{cases}$$
‘Two is a large number’

**Proposition:** There is a unique Nash-equilibrium in which both players set prices equal to marginal cost and obtain zero profits

\[(p^*_A, p^*_B) = (c, c)\]

**Proof:** Equilibrium: By decreasing the price below \(p^*\) a firm will incur losses. Increasing the price over \(p^*\) does not increase profits. (In equilibrium both firms play a weakly dominated strategy)

Uniqueness: No equilibrium can entail a \(p < 0\). Suppose there is an equilibrium with \(p > c\) for at least on firm. Then the rival firm deviates by undercutting (i.e. by setting \(p - \varepsilon\) where \(\varepsilon > 0\) is chosen arbitrarily small).

**Note:** In equilibrium firms are indifferent between their equilibrium strategy and any other strategy.
Profit function

The optimization problem is not well behaved.
reaction functions

Unless there is a price grid with minimal steps $\varepsilon$, the reaction functions are not defined. If there is such a grid, then a Nash-equilibria exist so that $(p_A^*, p_B^*) = (c + \varepsilon, c + \varepsilon)$.

‘reaction function’:

$$R_A(p_B) = \begin{cases} 
  p_B - \varepsilon & \text{if } p_B > c + \varepsilon \\
  c + \varepsilon & \text{if } p_B \leq c + \varepsilon 
\end{cases}$$

Since $R_A(p_1) \geq R_A(p_0)$ for $p_1 \geq p_0$ we have strategic complements.
Reaction Functions

\[ R_A, p_A \]

\[ p^m \]

\[ c \]

\[ R_B \]

\[ R_A \]

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Solving the Game Through Iterated Elimination of Dominated Strategies

**Without price grid**
If the strategy space is given by $[0, p^m]^2$ the set of rationalizable strategies is

$$p_A \in [c, p^m); \quad p_B \in [c, p^m).$$

**With a price grid**
If the strategy space is given by $\{0, \epsilon, 2\epsilon, 3\epsilon...N\epsilon = p^m\}^2$ the set of rationalizable strategies is

$$p_A \in \{c, c + \epsilon\}; \quad p_B \in \{c, c + \epsilon\}.$$
With Bertrand competition two firms are enough to obtain the same results as with perfect competition.

But how robust is this result?

**Richer strategy space:** ‘price matching’:

The firm sets a price and offers to match any lower price offered by a competitor.

This sounds good for customers. But any price–pair can be an equilibrium including the monopoly price!
Example

In 1992 a subsidiary of German Telecom ‘DeTeMedien’ started selling a telephone directory on CD-ROM for almost DM 4000 per copy. These directories contained: names, telephone numbers and addresses of all the people connected to the net in Germany. They were very convenient for businessmen. As this data was stored digitally anyway, there were little additional cost of production, distribution etc. estimated at DM 10 per CD. By October 1995 the price was still higher than DM 1500. Telecom was selling ‘a five-digit number of copies’. At that time TopWare appeared at the market as the only competitor. It digitalized ‘traditional’ telephone directories books by scanner, and offered the CD at the price of DM 50 selling 750 000 items, within a few weeks. Telecom responded by lowering the price to DM 90 and obtained a prelimentary court ruling preventing TopWare from selling, on the grounds that scanning of telephone directories is an infringement of copyrights.
With legal procedures underway, TopWare actualized its version of CD early 1996 by having the whole directory retyped by 600 typists in China. It offered the new version again for DM 90. Telecom opened a second case against TopWare in court, but lowered the price for its new version to DM 30 in March 1996. So, the price dropped from DM 1500 to DM 30 during 6 months. Hence, within six month the price dropped from DM 1500 to DM 30 — which is comparable to the competitive market for music CD.

Do the assumptions of Bertrand competition apply? What alternative strategies could have been adopted? Why have they not been chosen?
Stackelberg (1934)

Assumptions:

1. Risk neutral firms maximize profits.
2. Firms produce a homogeneous product.
3. All cost functions are known to all firms.
4. Firms meet once and cannot make binding agreements.
5. The leader $l$ selects his quantity (capacity): $q_l \in [0, D(0)]$.
6. The followers observe $q_l$ and select their quantities: $q_i \in [0, D(0)]$.
7. Price is chosen (by an auctioneer) to clear the market $D(p) = \sum q_i$.

Reinterpretation of quantities as capacities.
Normal Form of the Stackelberg Game:

**Players:** Firms $A$ (leader) and $B$ (follower)

**(pure) strategies:**

$$q_A \in [0, \bar{q}]; \quad Q_B : [0, \bar{q}] \rightarrow [0, \bar{q}]; \quad \bar{q} = D(0)$$

The strategy of the follower $B$ is rule for every possible initial choice of player $A$, hence a function.

**Pay-off functions:**

$$\pi_A(q_A, Q_B) = q_A \cdot P(q_A + Q_B(q_A)) - K_A(q_A)$$

$$\pi_B(q_A, Q_B) = Q_B(q_A) \cdot P(q_A + Q_B(q_A)) - K_B(Q_B(q_A))$$
The Solution

**Equilibrium** is a pair of strategies \((q_A^*, Q_B^*)\) which is mutually best response.

**Equilibrium-action:** In equilibrium we observe only two actions first: \((q_A^* \text{ then } Q_B^*(q_A^*))\).
The Stackelberg Follower, a Monopolist?

There exists a Nash-equilibrium in which $B$ obtains the monopoly profit, with

$$q_A = 0, \quad Q_B(q_A) = \begin{cases} q^M, & \text{if } q_A = 0 \\ D(0) - q_A, & \text{otherwise} \end{cases}$$

It is easy to check, that no one wants to deviate.
This solution is not convincing

**Problem**: The equilibrium is not convincing. It is based on empty threats in situations which are not reached in equilibrium. Suppose $A$ would chose the Cournot quantity $q^c_A$ (which is out of equilibrium) then it would not be in the interest of $B$ to play $D(0) - q^c_A$ but rather $q^c_B$. 
The Subgame Perfect Equilibrium SPB

To find the SPB, we start with the subgames after the choice of \( A \) (backward induction). Since the only remaining action is \( B \)'s choice of quantity, subgame-perfectness requires, that it is in \( B \)'s best interest to follow his strategy in every possible subgame, i.e. for every possible choice of \( q_A \). Hence we require:

\[
Q_B = \arg \max_{q_B} \pi_B(q_A, q_B), \forall q_A.
\]

which defines the Cournot reaction function \( Q_B^* \). The equilibrium strategy of \( A \) is found by ‘maximizing against \( B \)'s reaction function’:

\[
q_A^* = \arg \max_{q_A} \pi_A(q_A, Q_B^*(q_A))
\]
The Value of Commitment

To compare Cournot and Stackelberg note that

\[ q^*_A \text{ solves } \frac{\partial \pi_A(q^*_A, q_B)}{\partial q_A} + \frac{\partial \pi_A(q^*_A, q_B)}{\partial q_B} \cdot \frac{dQ^*_B(q^*_A)}{dq_A} = 0. \]

The second term indicates the strategic effect. Provided that quantities are strategic substitutes, it is positive. Hence \( q^*_A > q^*_A \).

1. Leader produces more than under Cournot competition and obtains higher profits.
2. Follower produces less and obtains lower profits.