An introduction to functional analysis

You will learn

- key concepts from functional analysis, which are useful for the theory of dynamic programming as well as in a variety of other contexts,
- definitions and examples for vector spaces, metric spaces, normed spaces, Banach spaces, Hilbert spaces
- definitions and examples for operators and functionals, including the conditional expectations operator and the lag operator
- definitions, examples and properties of compact metric spaces and contractions,
- Blackwells sufficient conditions for a contraction and
- why finite-dimensional spaces are fundamentally different from infinite-dimensional spaces

Sources

- Nancy L. Stokey and Robert E. Lucas, Jr., with Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989
- Books on functional analysis, e.g. Rudin or Dunford and Schwarz.

Motivation

- The generic Bellman equation is
  - for deterministic problems:

$$v(x) = \max_y \{F(x,y) + \beta v(y) \mid y \in \Gamma(x)\}$$

  - for stochastic problems:

$$v(x,z) = \max_y \{F(x,y,z) + \beta E[v(y,z') \mid z] \mid y \in \Gamma(x,z)\}$$

- The unknown object in these equations is not a number but an entire function, namely the value function. I.e., the Bellman equation is a **functional equation**.
- To study the properties of solutions to equations, in which the unknown object is a function, mathematicians have invented **functional analysis**. We need to study it.
- The key idea is to regard functions as elements in a **vector space** of functions, and to regard e.g. the right-hand side of the Bellman equation as an **operator** on this space. Functional analysis provides a theory for vector spaces and their operators.

Spaces

We first need to introduce a variety of spaces and related concepts:

- vector spaces
- metric spaces
- normed spaces
- Banach spaces
- Hilbert spaces

Vector Spaces

- **Definition 1** *A* **real vector space** X *is a set of elements (vectors), including a zero vector* $\mathbf{0}$*, together with two operations,*

  - *addition*
  - *scalar multiplication with real numbers,*

  obeying the usual rules.

- The "usual rules" for $x, y \in X$ and $\alpha, \beta \in \mathbf{R}$ are

$$\begin{aligned} x + y &= y + x \\ (x + y) + z &= x + (y + z) \\ \alpha(x + y) &= \alpha x + \alpha y \\ (\alpha + \beta)x &= \alpha x + \beta x \\ (\alpha\beta)x &= \alpha(\beta x) \\ x + \mathbf{0} &= x \\ 0x &= \mathbf{0} \\ 1x &= x \end{aligned}$$

Vector spaces: examples

- any $\mathbf{R}^n$, including $\mathbf{R}$ itself.
- Let S be any set. The set $B(S)$ of all bounded functions $f : S \to \mathbf{R}$ is a vector space with point-by-point addition and scalar multiplication.
- The set $C([a,b])$ of all continuous functions on the interval $[a,b]$.
- The set of all continuously differentiable functions $C^1([a,b])$ on the interval $[a,b]$.
- The set of all continuous functions $C(\mathbf{R})$ on $\mathbf{R}$.
- The set of all continuous bounded functions $C^b(\mathbf{R})$ on $\mathbf{R}$.
- The set l_0 of all real-valued sequences $(x_n)_{n=0}^{\infty}$.
- The set l_∞ of all **bounded** sequences

Not a vector space are

- $\mathbf{R}^n_+$, the subset of all $x \in \mathbf{R}^n$ with only nonnegative entries.

- The set of all concave functions on the interval $[a, b]$.
- The set of vectors with only rational numbers as entries.

Metric spaces

Definition 2 *A* **metric space** *is a set S together with a metric $d : S \times S \to \mathbf{R}$ such that for all $x, y, z \in S$,*

$$\begin{aligned} d(x,y) &\geq 0 \\ d(x,y) = 0 & \quad \textit{if and only if } x = y \\ d(x,y) &= d(y,x) \\ d(x,z) &\geq d(x,y) + d(y,z) \end{aligned}$$

The study of metric spaces or, more generally, of **topological spaces** is the realm of **topology**, another mathematical field.

Metric spaces: examples

- Any set of points S with the **discrete metric**: $d(x, y) = 1$, unless $x = y$, in which case $d(x, y) = 0$.
- Any subset of a metric space, together with the same metric.
- $\mathbf{R}^n$ together with $d(x, y) =\| x - y \|$.
- The rational numbers with $d(x, y) =| x - y |$.
- $\mathbf{R}^n_+$ together with $d(x, y) =\| x - y \|$.
- Not a metric space: $C(\mathbf{R})$ with

$$d(f, g) = \sup_{t\in\mathbf{R}} | f(t) - g(t) |$$

- A metric space: $C(\mathbf{R})$ with

$$d(f, g) = \min\{1; \sup_{t\in\mathbf{R}} | f(t) - g(t) |\} \tag{1}$$

Complete metric spaces

Let (S, d) be a metric space.

- **Definition 3** *A* **Cauchysequence** *is a sequence* $\{x_n\}_{n=1}^{\infty}$ *such that for every* $\epsilon > 0$*, there is a* N_ϵ*, so that* $d(x_n, x_m) < \epsilon$ *for all* $n, m \geq N_\epsilon$*.*

- **Definition 4** *A Cauchy sequence* **converges** *to* x*,*

$$x_n \to x$$

  if for every $\epsilon > 0$*, there is a* N_ϵ*, so that* $d(x_n, x) < \epsilon$ *for all* $n \geq N_\epsilon$*.*

- **Definition 5** *A metric space* (S, d) *is* **complete***, if every Cauchy sequence converges.*

Complete metric spaces: examples

- The rational numbers with $d(x, y) = \mid x - y \mid$ is not a complete metric space.
- The rational numbers (or any other space) with the discrete metric is a complete metric space.

Normed spaces and Banach spaces

- **Definition 6** *A* **normed vector space** *is a vector space X together with a norm $\|\cdot\|: X \to \mathbf{R}$ so that for all $x, y \in X$ and $\alpha \in \mathbf{R}$,*

$$\begin{aligned} \| x \| &\geq 0 \\ \| x \| = 0 & \quad \textit{if and only if } x = \mathbf{0} \\ \| \alpha x \| &= | \alpha | \| x \| \\ \| x + y \| &\leq \| x \| + \| y \| \end{aligned}$$

  The last inequality is called the **triangle inequality**

- A normed vector space is a metric space with

$$d(x, y) = \| x - y \|$$

- **Definition 7** *A* **Banach space** *is a complete normed vector space.*

Normed spaces and Banach spaces: examples

Suppose throughout that $-\infty < a < b < \infty$

- $\mathbf{R}^n$ with the usual norm is a Banach space.
- For $f \in B(S)$, the space of bounded functions on S into $\mathbf{R}$, define the **sup norm**

  $$|| f || = \sup_{s \in S} | f(s) | \tag{2}$$

  Then, $(B(S), || \cdot ||)$ is a Banach space.
- The space $(C([a,b]), || \cdot ||)$ of continuous functions on the interval $[a,b]$ with the sup norm is a Banach space.
- The space of concave, continuous functions on $[a,b]$ with $d(f,g) = || f - g ||$ is a complete metric space, but not a vector space, and thus certainly not a Banach space.
- $C(\mathbf{R})$ with the metric given in equation (1) is a complete metric space, but not a normed space.
- $C^b(\mathbf{R})$ with

  $$|| f || = \sup_{t \in \mathbf{R}} | f(t) |$$

is a Banach space.

- $(C^1([a,b]), || \cdot ||)$ with $|| \cdot ||$ given by (2) is a normed space, but not a Banach space.

- $(C^1([a,b]), || \cdot ||_{D1})$ with

$$|| f ||_{D1} = || f || + || f' ||$$

is a Banach space.

- The set l_∞ of all **bounded** sequences, i.e. for which

$$|| x ||_\infty = \sup_n | x_n | < \infty$$

together with this norm is a Banach space.

- Let $1 \leq p < \infty$. The set l_p of all real-valued sequences, so that

$$|| x ||_p = \left(\sum_{n=0}^{\infty} | x_n |^p \right)^{1/p} < \infty \qquad (3)$$

together with this norm is a Banach space. The proof that (4) satisfies the triangle inequality is called the **Minkowski inequality** and its proof is a bit tricky.

- A property, which is sometimes useful: let $p, q \geq 1$ such that
$$\frac{1}{p} + \frac{1}{q} = 1$$
Let $(x_n)_{n=0}^{\infty} \in l_p$ and $(y_n)_{n=0}^{\infty} \in l_q$. Then
$$\sum_{n=0}^{\infty} \mid x_n y_n \mid \leq \| x \|_p \| y \|_q$$
This is called the **Hölder inequality**.

- Let $1 \leq p < \infty$. The space $C([a, b])$ with
$$\| f \|_p = \left(\int_a^b \mid f(t) \mid^p \right)^{1/p} \tag{4}$$
is a normed space, but not a Banach space. Again, the triangle inequality is called the **Minkowski inequality** and its proof is a bit tricky. The Hölder inequality holds here too.

Completion

If a normed vector space is not a Banach space, one can embed it into a Banach space by "adding" all the "missing" limiting vectors of Cauchy sequences. This is similar to the construction of the real numbers from rational numbers. Examples:

- The completion of any Banach space is the space itself.
- The completion of $(C^1([a,b], || \cdot ||)$ with the norm given by (2) is the space $(C([a,b], || \cdot ||)$ of continuous functions with that norm.
- Let $1 \leq p < \infty$. The completion of $(C^p([a,b], || \cdot ||_p)$ is the Banach space $(L^p([a,b], || \cdot ||_p)$ of (equivalence classes of) Lebesgue-integrable functions on $[a,b]$, for which

$$|| \, f \, ||_p = \left(\int_a^b \mid f(t) \mid^p \right)^{1/p} < \infty$$

Hilbert spaces

- **Definition 8** *A* **scalar product** (x, y) *on a vector space is a mapping of two vectors into the real line, satisfying*

$$\begin{aligned}(x,x) &\geq 0 \\ (x,x)=0 & \quad \textit{if and only if } x=\mathbf{0} \\ (\alpha x, y) &= \alpha(x,y) \\ (x+y,z) &= (x,z)+(y,z) \\ (x,y) &= (y,x)\end{aligned}$$

- **Definition 9** *A* **Hilbert space** *is a vector space* X *with an inner product and a norm defined by*

$$|| \, x \, || = \sqrt{(x,x)} \tag{5}$$

so that $(X, || \cdot ||)$ *is a Banach space.*

The Cauchy-Schwarz inequality

$$\mid (x, y) \mid \leq \mid\mid x \mid\mid \mid\mid y \mid\mid$$

PROOF: Using just the rules about the inner product and the definition of the norm (5), rewrite

$$\begin{aligned} 0 &\leq (\frac{x}{\mid\mid x \mid\mid} - \frac{y}{\mid\mid y \mid\mid}, \frac{x}{\mid\mid x \mid\mid} - \frac{y}{\mid\mid y \mid\mid}) \\ &= (\frac{x}{\mid\mid x \mid\mid}, \frac{x}{\mid\mid x \mid\mid}) - 2(\frac{x}{\mid\mid x \mid\mid}, \frac{y}{\mid\mid y \mid\mid}) + (\frac{y}{\mid\mid y \mid\mid}, \frac{y}{\mid\mid y \mid\mid}) \end{aligned}$$

as

$$2\frac{(x, y)}{\mid\mid x \mid\mid \mid\mid y \mid\mid} \leq \frac{(x, x)}{\mid\mid x \mid\mid \mid\mid x \mid\mid} + \frac{(y, y)}{\mid\mid y \mid\mid \mid\mid y \mid\mid} = 2$$

Done. $\square$

Hilbert space: examples

- $\mathbf{R}^n$ with
$$(x, y) = x'y = \sum_{j=1}^{n} x_j y_j$$
is a Hilbert space.

- The space l_2 with
$$(x, y) = \sum_{n=1}^{\infty} x_n y_n$$
is a Hilbert space.

- The space $L_2([a, b])$ with
$$(f, g) = \int_a^b f(t) g(t)$$
is a Hilbert space.

Operators

We will now consider mappings from some space into others. Depending on the context, they will be called **operators**, **mappings**, **functionals** or simply **functions**. In particular, we will consider

- linear operators on vector spaces
- functionals on vector spaces
- continuous functions on metric spaces
- ... and study their properties on **compact** sets
- contraction mappings

Definitions

- A mapping T from one space into another is called an **operator**. We often write Tx instead of $T(x)$.
- Of particular importance for vector spaces are **linear operators**:

  Definition 10 *Let X, Y be two vector spaces. A* **linear operator** $T : X \to Y$ *is a mapping, satisfying*

  $$T(\alpha x + \beta z) = \alpha Tx + \beta Tz$$

  for all $\alpha, \beta \in \mathbf{R}$ and $x, z \in X$.

- **Definition 11** *Suppose that X, Y are normed spaces. The linear operator $T : X \to Y$ is* **continuous**, *if $\| x_n \| \to 0$ implies $\| Tx_n \| \to 0$.*

Examples for linear operators

- Let A be a $n \times m$ matrix. Then,

  $$y = Tx = Ax$$

  is a continuous linear operator $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$.

- Let $S = \{s^{(1)}, \ldots, s^{(n)}\}$ be a finite set of states.
  - Define a **Markov process** $(s_t)_{t=0}^{\infty}$ on S per **transition probabilities**

    $$\pi_{ij} = P(s_{t+1} = s^{(j)} \mid s_t = s^{(i)})$$

  - For any function $f : S \rightarrow \mathbf{R}$, one can calculate the **conditional expectation**

    $$f^e(s^{(i)}) = (E[f(s_{t+1}) \mid s_t = s^{(i)}) = \sum_{j=1}^{n} \pi_{ij} f(s^{(j)})$$

  - The **conditional expectations operator** $T : B(S) \rightarrow B(S)$ is the mapping defined by $Tf = f^e$.
  - One can write this operator in matrix notation. Identify f with the vector $[f(s^{(1)}), \ldots, f(s^{(n)})]'$ in $\mathbf{R}^n$. Let Π be the matrix with entries π_{ij}. Then,

    $$Tf = \Pi f$$

- Let $-\infty < a < b < \infty$ and let the **kernel** $k(x,y)$ be a continuous function of $[a,b] \times [a,b]$. Define the **integral operator** $T : C([a,b]) \to C([a,b])$ per

  $$Tf(x) = \int_a^b k(x,y) f(y) dy$$

  The operator is continuous with respect to the sup norm.

- Suppose in addition, that $k(x,\cdot)$ is a probability density,

  $$\text{for all } x\text{: } 1 = \int_a^b f(x,y) dy$$

  defining a Markov process on $[a,b]$. Then,

  $$Tf(x) = E[f(x') \mid x]$$

  is again the **conditional expectations operator**.

- The **lag-operator** $T : l_p \to l_p$ defined by

  $$(Tx)_n = \begin{cases} x_{n-1} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

  The operator is continuous.

Functionals

- **Definition 12** *Let X be a vector space. A linear operator $T : X \to \mathbf{R}$, mapping X into real numbers, is called a* **functional**

- "Functionals" have given "functional analysis" its name and are its main building block.

- **Definition 13** *Let X be a normed vector space. The* **dual space** *X' is the space of all continuous linear functionals $T : X \to \mathbf{R}$.*

Functionals: examples

- - Let $y \in \mathbf{R}^n$. It can be understood as a functional T on $x \in \mathbf{R}^n$ per

    $$Tx = y'x$$

  - The dual space of $\mathbf{R}^n$ is $\mathbf{R}^n$.

- - Let $1 \le p, q$ with

    $$\frac{1}{p} + \frac{1}{q} = 1$$

    Let $y \in l_q$. It can be understood as a functional T on l_p per

    $$Tx = \sum_{n=0}^{\infty} x_n y_n$$

    Hölders inequality shows T to be continuous.

  - The dual space for l_p is l_q.
  - In particular, $l_2' = l_2$.

- This is general: the dual space of a Hilbert space is the Hilbert space itself via interpreting $y \in H$ as the functional T defined as

  $$Tx = (x, y)$$

- - Let $a \leq c \leq b$. $T : C([a, b]) \to \mathbf{R}$, defined by $T_c f = f(c)$ is a continuous functional, called the **evaluation functional**.
  - The dual space of $C([a, b])$ is the space $M([a, b])$ of **signed measures** on $[a, b]$. Defining this in greater detail requires knowledge of **measure theory**, another mathematical field.

Compactness

Let (S, d) be a metric space.

- **Definition 14** *A subset A of S is called* **closed***, if whenever $x_n \in A$ and $x_n \to z$, then $z \in A$.*

- **Definition 15** *A subset A of S is called* **open***, if its* **complement** *S A, i.e., the set of all points in S but not in A, is closed.*

- **Definition 16** *A subset A of a metric space is called* **bounded***, if there is some $M < \infty$ so that $d(x, y) \leq M$ for all $x, y \in A$.*

- **Definition 17** *Let (S, d) be a metric space. A function $f : S \to \mathbf{R}$ is called* **continuous***, if $x_n \to x$ implies $f(x_n) \to f(x)$.*

- **Definition 18** *A subset A of a metric space is called* **compact***, if every sequence has a convergent subsequence.*

- Special case: $A = S$. In that case, the space is called a compact metric space.

- A compact metric space is always complete.

- **Theorem 1** *Let* (S, d) *be a compact metric space and let* $f : S \to \mathbf{R}$ *be continuous. Then* f *achieves its maximum on* S*, i.e. there is a point* $\bar{x} \in S$ *so that*

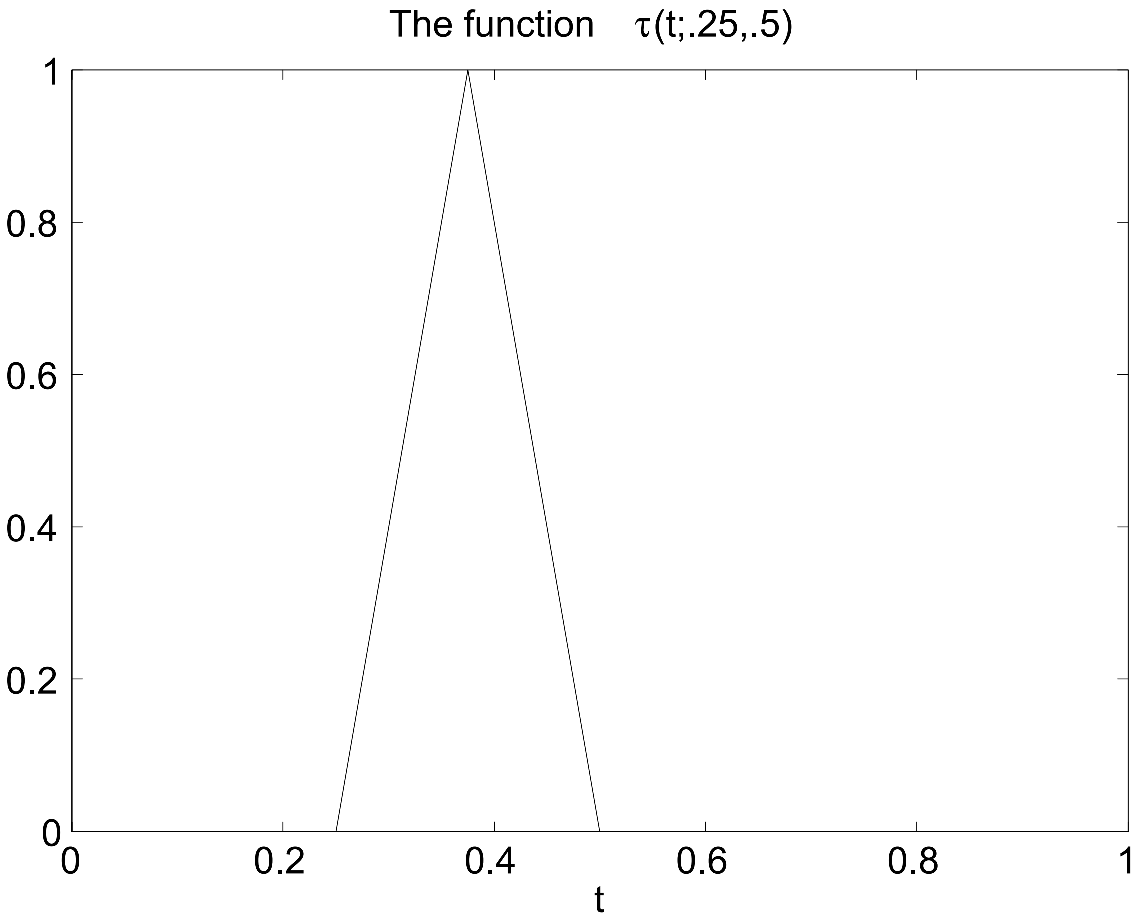
$$f(\bar{x}) = \max_x f(x)$$

Examples

- Every compact subset of a metric space is closed and bounded. The reverse is generally not true.
- Every closed, bounded subset of $\mathbf{R}^n$ is compact.
- Consider the Banach space $(C([a,b]), ||\cdot||)$.
  - Let $\mathcal{B}$ be the **unit ball**, consisting out of all $f \in C([a,b])$ with norm less or equal to 1. Then B is closed and bounded, but not compact.
  - This can be seen as follows. Suppose w.l.o.g., that $a = 0$, $b = 1$. For any two points $c < d$, define the tent function

$$\tau(t; c, d) = \begin{cases} 0 & \text{if } t < c \\ 2\frac{t-c}{d-c} & \text{if } c \leq t < \frac{c+d}{2} \\ 2\frac{d-t}{d-c} & \text{if } \frac{c+d}{2} \leq t \leq d \\ 0 & \text{if } t > 0 \end{cases}$$

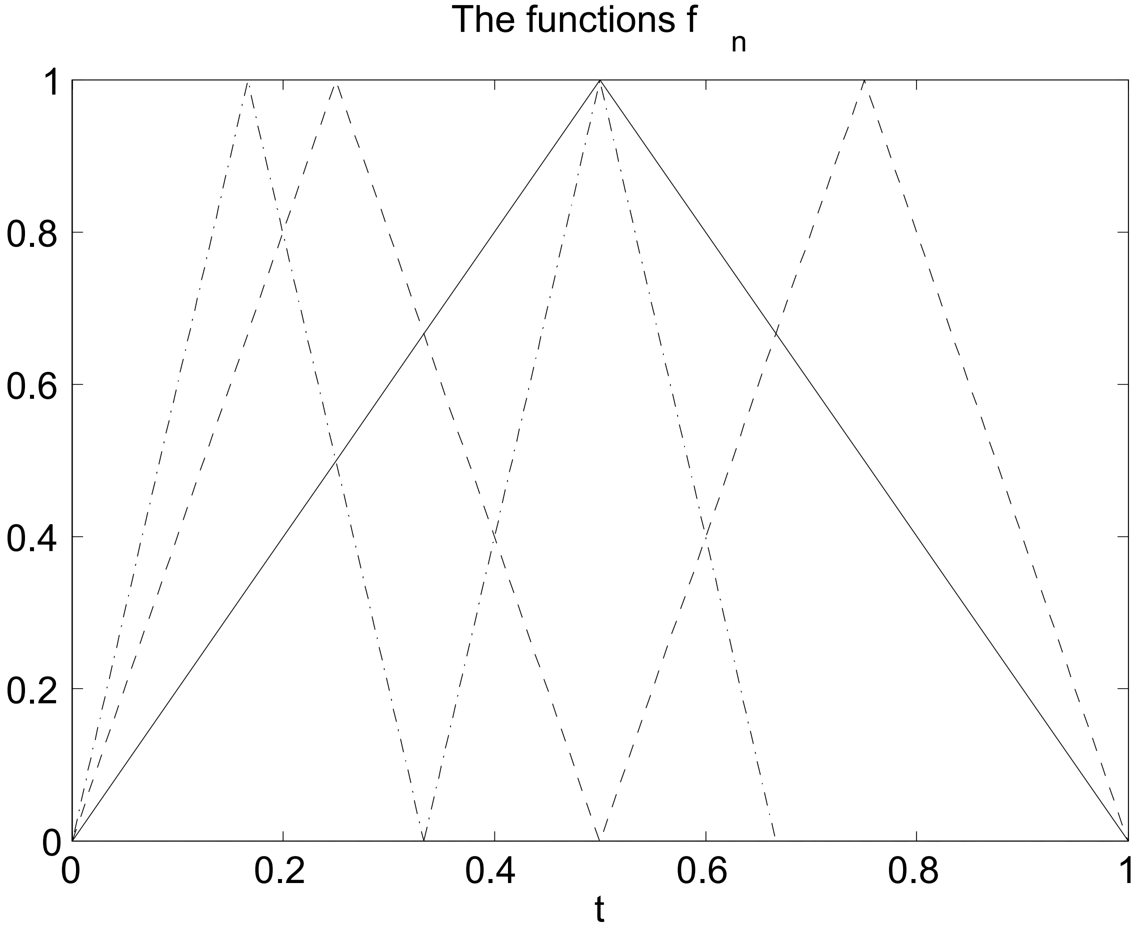
Note that $\tau(\cdot; a, b)$ is continuous, has a minimum of zero and a maximum of 1.



- Define

$$\begin{aligned} f_1(t) &= \tau(t; 0, 1) \\ f_2(t) &= \tau(t; 0, \frac{1}{2}) \\ f_3(t) &= \tau(t; \frac{1}{2}, 1) \\ f_4(t) &= \tau(t; 0, \frac{1}{3}) \\ f_5(t) &= \tau(t; \frac{1}{3}, \frac{2}{3}) \end{aligned}$$

etc.. Then, $(f_n)_{n=0}^{\infty}$ contains no convergent subsequence.



- **Definition 19** *A subset A of $C([a,b])$ is called* **equicontinuous***, if for all $\epsilon > 0$, there is $\delta > 0$, so that $\mid t - s \mid < \delta$ implies that $\mid f(t) - f(s) \mid < \epsilon$ for all $f \in A$.*
- **Theorem 2 [Arzela-Ascoli]** *Let A be a closed, bounded, equicontinuous subset of $C([a,b])$. Then A is compact.*

Finite versus infinite dimensions

- **Definition 20** *A vector space is* **finite dimensional**, *if there are finitely many vectors* $x_1, \dots, x_n$, *so that every vector* x *can be written as a* **linear combination**

  $$x = \sum_{j=1}^{n} \alpha_j x_j$$

  for some real numbers $\alpha_1, \dots, \alpha_n$. *If not, the space is* **infinite dimensional**.

- Except for $\mathbf{R}^n$ or for $B(S)$, where S contains finitely many points, all our examples for vector spaces are infinite-dimensional.

- Suppose that $(X, || \cdot ||)$ is a Banach space. Let $\mathcal{B}$ be the unit ball, consisting out of all $x \in X$ with $|| x || \leq 1$.

  Theorem 3 *If* $\mathcal{B}$ *is compact, then* X *is finite dimensional.*

- This theorem shows that finite dimensional spaces and infinite-dimensional spaces are fundamentally different.

Contraction mappings

- **Definition 21** *Let* (S, d) *be a metric space and* $T : S \to S$ *be an operator, mapping* T *into itself.* T *is called a* **contraction mapping** *(with* **modulus** β*), if for some* $\beta \in (0, 1)$*,*

  $$d(Tx, Ty) \leq \beta d(x, y)$$

  for all $x, y \in S$*.*

- **Theorem 4 [Contraction mapping theorem]** *Let* (S, d) *be a complete metric space and* $T : S \to S$ *be a contraction mapping with modulus* β*. Then,*

  - T *has exactly one* **fixed point** v *in* S*,*

    $$Tv = v$$

  - *For any* $v_0 \in S$*, and all* $n = 0, 1, 2, \ldots$*,*

    $$d(T^n v_0, v) \leq \beta^n d(v_0, v)$$

- **Theorem 5 [Blackwell's sufficient conditions for a contraction]** *Let* $B(X)$ *be the Banach space of bounded functions* $f : B \to \mathbf{R}$ *with the sup norm (2) on some set* X*. Let* $T : B(X) \to B(X)$ *be an operator satisfying*

- **[monotonicity:]** *$f, g \in B(X)$ and $f(x) \leq g(x)$ for $x \in X$ implies*

  $$(Tf)(x) \leq (Tg)(x)$$

  for all $x \in X$.

- **[discounting:]** *there exists some $\beta \in (0,1)$, such that*

  $$(T(f+a))(x) \leq (Tf)(x) + \beta a$$

  for all $f \in B(X)$, $a \geq 0$.

Then T is a contraction mapping with modulus β.