The Absent–Minded Centipede

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December 2, 1996

Abstract

In this note we apply the notion of absent–mindedness (see Piccione and Rubinstein, 1994), which is a form of imperfect recall, to Rosenthal's (1981) centipede game. We show that for standard versions of the centipede game a subgame perfect equilibrium exists in which play is continued almost to the end if one player is known to be absent–minded. In fact, it is sufficient that one player is known to be absent–minded with sufficiently high probability.

JEL– classification number: C72.

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1 Introduction

The centipede game (Rosenthal, 1981) is one of those games in which the game theoretic solution is miles away from intuition and experimental evidence. Inspired by the recent discussion on games with imperfect recall (Piccione and Rubinstein, 1994, and Aumann et al., 1996) we want to point out the effects imperfect recall has in the centipede game. Instead of assuming that one player may be 'irrational' (as in Kreps et al., 1982)[1], we assume that one player becomes *absent–minded* in the course of the game. A player is called absent-minded if he cannot remember whether he has visited a decision node before. In particular, we want to assume that a player might forget whether he has already passed his second last decision node – maybe because he miscounted the number of decision nodes. When counting to, say, one hundred this can certainly happen. Arriving at the last node the player does not know this for sure. He thinks it possible that he has one more to go.

We show that with absent–mindedness for standard versions of the centipede game there exists a subgame perfect equilibrium in which the game is continued (almost) to the end.

2 An example

Consider the centipede game in Figure 1.

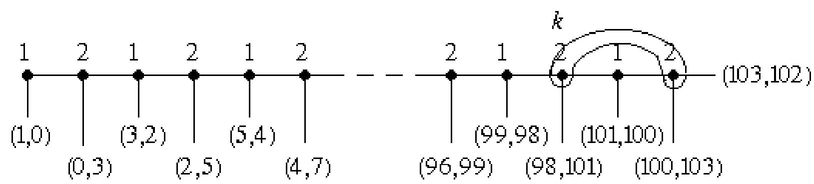

Figure 1: An example

The only difference to the normal centipede game is the information set containing the last two decision nodes of player 2. Obviously, player 2 is

[1] Kreps (1990) and McKelvey and Palfrey (1992) apply the 'gang of four' approach to the centipede game.

absent–minded as he cannot remember whether he has made a decision at his second last decision node before.

In the standard version of the centipede game (without 2's information set) there exists a unique subgame perfect equilibrium, in which both players choose D always. There are some imperfect Nash equilibria as well, but all Nash equilibria begin with player 1 choosing D at the first node.

Consider now the subgame Γ^k beginning with node k of the modified game.[2] Since player 2 has only one information set, he has two pure strategies, C and D. There are two pure strategy equilibria. In the D–equilibrium both players choose D. To check whether (C, C) is an equilibrium we have to specify the beliefs of player 2 about where he is in his information set. Let $\mu(k)$ denote 2's belief that he is in k. If both players use C, it seems uncontroversial to suppose that $\mu(k) = \mu(k+2) = 1/2$. Given these beliefs (C, C) is a Nash equilibrium of Γ^k. In fact, (C, C) is the payoff dominant Nash equilibrium.

Hence, the whole game has also two pure subgame perfect equilibria. The first is the standard centipede equilibrium with each player choosing D at every node. However, with absent-mindedness there exists another, payoff–dominant, subgame perfect equilibrium, namely always choosing C. Note that the same results would hold, a fortiori, if player 2 could not distinguish between his last $d > 2$ nodes or if player 1 were absent–minded too.

3 The general case

The general centipede game is a finite move game in which two players alternate in deciding whether to achieve a short run gain by terminating the game or whether to let the game continue, which results in long run gain for both players. Let $i = 1, ..., n$ denote the nodes of the game. At odd nodes it is player 1's turn, at even nodes player 2's. Let (a_i, b_i) denote the payoff of player 1 and 2, respectively, when the game is terminated at node i. (a_{n+1}, b_{n+1}) denote the payoffs when the game is continued all the way to the end. A general property of the centipede game is that a player risks a short run loss when continuing the game, which is, however, more than offset if the game is continued for two steps, that is, for i odd

$$a_{i+2} > a_i > a_{i+1}$$

[2] Γ^k is a proper subgame. See the definition of a subgame for games with imperfect recall in Selten (1975).

and similarly for b_i if i even. To simplify notation we will assume that the gains $\Delta^+ := a_{i+2} - a_{i+1}$ and the losses $\Delta^- := a_i - a_{i+1}$ are constant and the same for both players.

The generalization of the above argument is the following. Without loss of generality, let player 2 be the absent–minded player. Further, let d denote the *degree of absent–mindedness* measured by the number of nodes among which player 2 cannot distinguish (beginning with node k, where k is even). If the equilibrium payoffs Π_1^k and Π_2^k in the subgame Γ^k satisfy $(\Pi_1^k, \Pi_2^k) \geq (a_{k-1}, b_{k-2})$, then there exists a subgame perfect equilibrium in which both players cooperate at least until node k. Of course, the existence of such equilibria depends on Δ^+ and Δ^-.

The equilibrium in the subgame can be in pure or in mixed strategies. In Proposition 1 we will first consider the case of pure strategies.

Proposition 1 *A Pareto efficient subgame perfect equilibrium in which both players use C at all nodes exists iff $d \geq \frac{\Delta^+ - \Delta^-}{\Delta^+ + \Delta^-}$. For $d = 2$ this requires $\Delta^+ \geq 3\Delta^-$.*

Proof Suppose player 2's absent–mindedness of degree d begins in node k. Given that player 1 continues always player 2's payoff from playing C is

$$\Pi_2^k(C) = b_k + (d-1)\Delta^+ - d\Delta^-.$$

The ex-ante payoff from playing D is b_k. However, as Piccione and Rubinstein (1994) argue, a player who planned to use C and consequently thinks he is in all nodes $i = k, k+2, k+4, ..., n$ with equal probability will expect to receive a payoff from playing D of

$$\Pi_2^k(D) = \frac{1}{d}\left(db_k + \frac{d(d-1)}{2}(\Delta^+ - \Delta^-)\right).$$

Thus, $\Pi_2^k(C) \geq \Pi_2^k(D)$ iff $d \geq \frac{\Delta^+ + \Delta^-}{\Delta^+ - \Delta^-}$.

Given player 2's strategy of continuing player 1's best response is to continue as well.

Since the game is one of imperfect recall, there is the possibility that a behavioral strategy does better than either pure strategy. Suppose this were the case. Player 2 would only randomize between C and D, if

$$\sum_{j=0}^{d-1} \mu(k+2j) b_{k+2j} \geq b_{n+1}. \qquad (1)$$

As shown above

$$\Pi_2^k(C) = b_{n+1} \geq \frac{1}{d}\sum_{j=0}^{d-1} b_{k+2j}.$$

Since the payoff sequence b_i with $i = k, k+2, ..., n$ is increasing, (1) can be satisfied only if $\mu(\cdot)$ puts more weight on later nodes (relative to $\frac{1}{d}$). However, if either player uses a non–degenerate behavioral strategy, a consistent belief $\mu(\cdot)$ must put more weight on early nodes.$\square$

We have established that for all (long enough) centipede games there exists a degree of absent–mindedness such that there is a subgame perfect equilibrium in which both players continue until the end of the game. For $d = 2$ the condition $\Delta^+ \geq 3\Delta^-$ is satisfied in the example above. The condition can be weakened if mixed equilibria of Γ^k are considered as well.

In the following we will specify conditions under which a mixed equilibrium exists for the case of $d = 2$. With payoff configurations for which mixed equilibria exist the endgame of our centipede has the structure of the "absent–minded driver" game. Piccione and Rubinstein (1994) argue that there is a time–inconsistency problem in that a player would want to deviate from his ex ante optimal decision during the course of the game.[3] We do not want to get into this debate but will instead present first the analysis with ex ante (or planning stage) optimal decisions[4] and show later how this has to be modified if one follows the arguments by Piccione and Rubinstein (1994).

Let p (q) denote player 2's (player 1's) behavioral strategy of choosing C. The payoff of player 1 given that he has reached node $n-1$ are then given by

$$\Pi_1^{n-1}(p,q) = (1-q)a_{n-1} + q(1-p)a_n + qpa_{n+1}. \tag{2}$$

Player 2's payoff before he reaches the information set $I = \{n-2, n\}$ is

$$\Pi_2^I(p,q) = (1-p)b_{n-2} + p(1-q)b_{n-1} + p(1-p)qb_n + p^2qb_{n+1}. \tag{3}$$

Lemma 1 *An (ex ante optimal) equilibrium of the subgame Γ^{n-2} in which at least one player uses a non–degenerate behavioral strategy exists iff $\Delta^+ \geq 2\Delta^-$.*

[3]This claim is strongly disputed by Aumann et al. (1996).

[4]The same solution holds true if one follows the approach by Auman et al. (1996) or Piccione and Rubinstein's (1994) multi-self approach.

Proof Since player 1's payoff is linear in p, he will choose C only if $p \geq \frac{a_{n-1}-a_n}{a_{n+1}-a_n} = \frac{\Delta^-}{\Delta^+}$ and be indifferent between C and D iff $p = \frac{\Delta^-}{\Delta^+}$.

Consider first the case that $0 < q < 1$. Hence, $p^* = \frac{\Delta^-}{\Delta^+}$. Maximizing $\Pi_2(p,q)$ with respect to p yields

$$p^* = \frac{q(b_n - b_{n-1}) - (b_{n-2} - b_{n-1})}{2q\,(b_n - b_{n+1})} = \frac{q\Delta^+ - \Delta^-}{2q\Delta^-}. \tag{4}$$

Substituting $\frac{\Delta^-}{\Delta^+}$ for p^* in (4)and solving for q yields

$$q^* = \frac{\Delta^+\Delta^-}{(\Delta^+)^2 - 2(\Delta^-)^2}.$$

Thus, $q^* < 1$ iff $\Delta^+ > 2\Delta^-$.

Next, consider the case of $q^{**} = 1$. Then, (4) reduces to

$$p^{**} = \frac{\Delta^+ - \Delta^-}{2\Delta^-}. \tag{5}$$

Note that $p^{**} < 1$ implies $\Delta^+ < 3\Delta^-$.

As shown above $q = 1$ is optimal for player 1 iff $p \geq \frac{\Delta^-}{\Delta^+}$. Combining this with (5) shows that an equilibrium with $q = 1$ exists only if $\Delta^+ \geq 2\Delta^-$.□

To summarize, the lemma shows that for $\Delta^+ > 2\Delta^-$ an equilibrium of Γ^k exists in which both players randomize. For $2\Delta^- \leq \Delta^+ < 3\Delta^-$ there is an equilibrium in which only player 2 randomizes. Finally, recall from Proposition 1 that for $\Delta^+ \geq 3\Delta^-$ an equilibrium exists in which both players play C with probability one.

Proposition 2 *For $\Delta^+ \geq 2\Delta^-$ there exists a subgame perfect equilibrium in which both players cooperate at least until node $n-2$.*

Proof We need to show that no player has an incentive to deviate from C before reaching the subgame Γ^{n-2}. This requires that $\Pi_1^{n-3}(p,q) \geq a_{n-3}$ and $\Pi_2^I(p,q) \geq b_{n-4}$.

For $\Delta^+ \geq 3\Delta^-$ the statement follows directly from Proposition 1. For $2\Delta^- \leq \Delta^+ < 3\Delta^-$ consider the equilibrium in which only player 2 randomizes ($q = 1$). We will show first that $\Pi_1^{n-3}(p^{**}, 1) \geq a_{n-3}$.

Note that since $q = 1$ is optimal in node $n-1$,

$$\Pi_1^{n-1}(p^{**}, 1) \geq a_{n-1} = a_{n-3} - \Delta^- + \Delta^+.$$

Since

$$\begin{aligned}\Pi_1^{n-3}(p^{**},1) &= (1-p^{**})(a_{n-3}-\Delta^-)+p^{**}\Pi_1^{n-1}(p^{**},1)\\ &\geq a_{n-3}-(1-p^{**})\Delta^- + p^{**}(\Delta^+-\Delta^-),\end{aligned}$$

it suffices to show that

$$a_{n-3}-(1-p^{**})\Delta^- + p^{**}(\Delta^+-\Delta^-) \geq a_{n-3}$$

or

$$p^{**}\Delta^+ \geq \Delta^-,$$

which is always satisfied as $p^{**} \geq \frac{\Delta^-}{\Delta^+}$.

To see that $\Pi_2^I(p^{**},1) \geq b_{n-4}$ note that in this equilibrium player 2 receives a payoff which is a convex combination of b_{n-2}, b_n, and b_{n+1}, all of which are larger than b_{n-4} if $\Delta^+ \geq 2\Delta^-$.□

If we require strategies to be time consistent in the sense of Piccione and Rubinstein (1994, section 6), we find similar to Lemma 1 that a subgame perfect equilibrium exists in which both players continue at least until node $n-2$. However, the condition on payoffs is stricter than in the previous case, namely, $\Delta^+ \geq (\frac{1}{2}+\frac{1}{2}\sqrt{13})\Delta^- \simeq 2.308\Delta^-$. Thus, for $2\Delta^- \leq \Delta^+ < (\frac{1}{2}+\frac{1}{2}\sqrt{13})\Delta^-$ it matters whether one follows the approach advocated by Piccione and Rubinstein (1994) or that by Aumann et al. (1996).

4 Absent–mindedness with positive probability

Following the idea of Kreps et al. (1982) an obvious question to ask is whether it is sufficient that one of the players is known to become absent–minded with some very small probability. This is not the case, however. The reason is that in contrast to the "irrational type" of Kreps et al. an absent–minded player is not committed to continue the game.

Nevertheless, we will show that our previous results continue to hold with some modifications even if the probability γ that player 2 is absent–minded is less than one. We are going to consider only the case of ex ante optimality and we restrict ourselves as in the previous section to the case of $d = 2$. The results could be generalized but this would not add to the content of the paper.

Given this setup we have to distinguish between player 2 being absent–minded or not. If he is not absent–minded (this happens with probability

$1-\gamma$), he will definitely choose D at the last node. Given he is absent–minded he will choose to continue with some probability p. The payoff of player 1 given his strategy q to continue at the second last node is given by

$$\Pi_1^{n-1} = (1-q)a_{n-1} + q((1-\gamma)a_n + \gamma((1-p)a_n + pa_{n+1})). \qquad (6)$$

Note that for $\gamma = 1$ (6) reduces to (2).

Proposition 3 *If $\gamma \geq \frac{2\Delta^-}{(\Delta^+)^2 - \Delta^-\Delta^+}$ then there exists a perfect Bayesian equilibrium in which both players continue at least until node $n-2$.*

Proof Let us first analyze the optimal strategy of player 1 who maximizes (6) with regard to q. He will strictly prefer C if $p > \frac{\Delta^-}{\gamma\Delta^+}$ and he will be indifferent between C and D if $p = \frac{\Delta^-}{\gamma\Delta^+}$.

Consider first $0 < q \leq 1$. If player 2 is absent–minded, he will maximize $\Pi_2^I(p,q)$ as given by (3). Because this optimization is not affected by the probability that he is absent–minded (he knows that he is), we arrive at the same solution $p^* = \frac{q\Delta^+ - \Delta^-}{2q\Delta^-}$ as in (4). As p^* increases in q, there is an equilibrium with $p^*, q^* > 0$ only if there is one for $q = 1$. Thus, for an equilibrium of Γ^{n-2} with $p^*, q^* > 0$ to exist p^* must satisfy $p^* = \frac{\Delta^+ - \Delta^-}{2\Delta^-} \geq \frac{\Delta^-}{\gamma\Delta^+}$,which yields

$$\gamma \geq \frac{2(\Delta^-)^2}{(\Delta^+)^2 - \Delta^-\Delta^+}$$

as stated in the Proposition. The remainder of the proof follows from the same reasoning as applied in the proof of Proposition 2.□

Thus, if the potential gain Δ^+ is sufficiently higher than the (sure) loss Δ^-, player 1 will play C even if player 2 is not absent–minded with high probability.

5 Conclusions

A little bit of absent–mindedness can go a long way in the centipede game. In fact, we showed in this note that if one player is known to be absent–minded with sufficiently high probability, then there is a perfect equilibrium in which both players continue almost to the end.

How do these results generalize to related games? For simplicity we used a centipede game with constant gains and losses. Obviously, the results hold true for more general formulations if the gains are only large enough relative to the losses. Currently we are working on applications of similar ideas to repeated games. Preliminary results indicate that cooperation in the finitely repeated prisoners' dilemma can be justified by absent–mindedness under certain conditions.

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