Commitment, Trembling Hand Imperfection and Observability in Games[1]

Brigitte Adolph

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Institut f. Wirtschaftstheorie I
Humboldt–Universität zu Berlin
Spandauer Str. 1
101 78 Berlin
Germany

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Abstract

In an important contribution Bagwell [1995] showed that the value of commitment tends to vanish if the observability of commitments is subject to an arbitrarily small distortion, due to the possibility of misunderstanding or communication error. Bagwell's observation calls into question the many stage games that have been exceedingly popular in economics, especially in theoretical industrial organization.

The present paper contributes to assess the robustness of Bagwell's result. We add other distortions to Bagwell model, such as "trembles" in players' execution of actions. We show that the unique pure strategy equilibrium of the game converges to the unique equilibrium outcome of the simultaneous move game with perfect observability if the noise associated with the observation of the leader's choice is small relative to the probability of trembles or if there are many other such imperfections. These results suggest that Bagwell's result is driven by his exclusive consideration of a particular distortion, and is not robust in the face of additional plausible distortions.

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JEL classifications: D 43, D 82, D 72.

1 Introduction

Many games in economics are sequential games where players make decisions in a given sequence. These decision are usually taken to be irreversible; hence, some ability to make reliable commitments is assumed. Examples abound especially in the field of theoretical industrial organization. The simplest cases are games with a Stackelberg "leader" and "follower" structure, and the many simultaneous move stage games that have been exceedingly popular in economics.

As was already pointed out by Schelling [1960] the ability to make irreversible commitments in and by itself has no value. Assuming simple games of complete information he showed that the power to make commitmentments is completely ineffective if these commitments cannot be observed by rival players.

Recently Bagwell [1995] further advanced our understanding of the role of observability in commitment games. Assuming a simple game of complete information Bagwell showed that commitment already becomes completely ineffective if the observation of commitments at earlier stages of the game is subject to a arbitrarily small noise, due to the possibility of misunderstanding or communication error.[1]

Bagwell's result calls into question the use of the exceedingly popular stage games in economics. However, before disposing of the child together with the bathwater one should explore variations of Bagwell's setup and assess the robustness of Bagwell result.

In a first assessment of the robustness of Bagwell's result Hurkens and van Damme [1994] argued that it is the focus on pure strategy equilibria that drives Bagwell's result. Under certain regularity assumptions Bagwell's game has also mixed strategy equilibria of which one preserves the value of commitment and converges to the unique equilibrium outcome of the simultaneous move game with perfect observability if the distortion tends to vanish. Therefore, if one has a convincing reason to favor that particular equilibrium, the value of commitment is restored. Hurkens and van Damme propose a new equilibrium selection principle that achieves precisely this goal.

However, this argument is not entirely convincing. Given their regularity

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[1] Güth, Kirchsteiger and Ritzberger [1996] confirmed that Bagwell's result also holds in games of complete information with n "leaders" and m "followers", where is n and m may be greater than 1.

assumption, each pure strategy equilibrium is strict, whereas mixed strategy equilibria are necessarily weak. Therefore, all standard equilibrium selection principles select the equilibrium in which the first mover advantage has vanished.

The present paper takes a different approach to rescue the value of commitment.[2] We assert that Bagwell's result is driven by the exclusion of other informational imperfections, such as "trembles" in players' execution of actions or payoff uncertainty. We add these distortions to Bagwell's model and show that the game then has an unique pure strategy equilibrium that converges to the unique equilibrium outcome of the simultaneous move game with perfect observability, if the noise associated with communication error is is small relative to the probability of trembles or if there are many kinds of imperfections.

2 An Example

First, consider a simple 2×2 game, in which there are two players who each choose one of two actions, C and S. The payoff matrix is

	S	C
S	2,1	0,0
C	3,2	1,3

Figure 1: Payoffs (player 1, player 2).

Suppose, player 1, the "Stackelberg–leader", moves first, and his action is perfectly observed by player 2. The unique subgame perfect equilibrium outcome then is (S, S) to which we refer as to the "Stackelberg outcome". Of course, this is also the unique sequential equilibrium and it is trembling hand perfect.

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[2] Another possible escape route — incomplete information — was explored by Adolph and Wolfstetter [1996]. in the framework of the well–known chain store game by Kreps and Wilson.

Consider a perturbed version of this game, and first assume that the possibility of misunderstanding or communication error is the only imperfection, as in Bagwell. Then, the unique pure strategy equilibrium outcome corresponds to the unique Nash equilibrium outcome (C,C) of the simultaneous–move game which we call the "Cournot outcome". For, suppose the observed signal deviates from the assumed equilibrium action, then player 2 has to conclude that this is due to a communication error — there is no other distortion. Player 2 thus completely ignores the observed signal. He always chooses his best response against $1's$ equilibrium action — no matter which signal he observes. Consequently, player 1 loses his first mover advantage and the "Cournot outcome" (C,C) is the unique pure strategy (Nash) equilibrium outcome.

However, suppose player 1 might make a mistake in the execution of the preferred action. Then, an observed deviation from player $1's$ equilibrium action is not necessarily due to a communication error. If player 2 believes that trembles are more likely than communication error, he will attach probability 1/2 or more to the event that the observed signal coincides with $1's$ actually executed action. In that case, in the simple 2×2 game, player 2 plays the Stackelberg–follower equilibrium strategy and chooses S if S was observed and C if C was observed.

3 Signal and Trembling Hand Imperfection

Consider a sequential game where player 1, the Stackelberg–leader, has the finite action space X with elements x, and player 2, the Stackelberg–follower, the finite action space Y with elements y.

Assumption 1 (Regularity) *Let* $u_i : X \times Y \to \mathbb{R}$ *denote the payoff functions of players* $i = 1, 2$. *We assume:*

$$u_i(x,y) \neq u_i(\hat{x},\hat{y}) \quad \forall (x,y) \neq (\hat{x},\hat{y}). \tag{1}$$

There are two kinds of imperfections. Due to trembles, player $1's$ actual choice x_1 is a realization of the random variable X_1 and differs with positive probability from his equilibrium action x_0. Player 2 neither observes x_0 nor x_1. Instead, he observes the realization x_2 of the random variable X_2.

The considered stochastic process $x_0 \rightarrow X_1 \rightarrow X_2$ is a Markov chain, with non stationary transition probabilities

$$\begin{aligned} p_{ij}^{10} &:= \Pr\{X_1 = i \mid x_0 = j\} > 0 \quad \forall (i,j) \in X^2, & (2)\\ p_{ij}^{21} &:= \Pr\{X_2 = i \mid x_1 = j\} \\ &= \Pr(X_2 = i \mid x_1 = j, x_0 = h\} > 0 \quad \forall\, (i,j,h) \in X^3. & (3) \end{aligned}$$

By (2)–(3), player 1 chooses any action $x_1 \in X$ with positive probability — given his equilibrium action x_0, and any signal $x_2 \in X$ can result from any $x_1 \in X$.

In this game, a pure strategy for player 1 is simply an action $x \in X$, and a pure strategy for player 2 is a function ω from signals X_2 to actions, so that the action $y \in Y$ taken when signal x_2 is observed is $y = \omega(x_2)$.

In order to derive the equilibrium outcome it is useful to define the best–response correspondences of the associated game without any imperfection. Let $R_2(x)$ denote the set of y values that maximize $u_2(x,y)$ over player $2's$ action space Y, and define $R_1(y)$ similar for Player 1. Notice that $R_1(y)$ and $R_2(x)$ are single valued by the regularity assumption (1).

By the regularity assumption (1), player 2 completely ignores the signal X_2, and always chooses his best respond against $1's$ equilibrium action x_0 if he concludes that the payoff relevant x_1 takes almost with certainty the same value as x_0 — no matter which signal x_2 is observed. In this case player $2's$ equilibrium strategy is $\omega^*(x_2) = R_2(x_0)$ for all $x_2 \in X$. However, player 2 reacts to the observed signal just as the "Stackelberg follower" of the associated game without any imperfection, and $\omega^*(x_2) = R_2(x_2)$ for all $x_2 \in X$, if he instead concludes that the payoff relevant x_1 coincides always almost with certainty with the signal X_2.

Player 2 uses all available information. He observes x_2, and he can infer $1's$ equilibrium action x_0. Therefore

Lemma 1 *There exists an $\delta > 0$ such that*

(i)

$$\Pr\{X_1 = k \mid x_2 = i, x_0 = k\} \geq 1 - \delta \quad \forall i \in X, \tag{4}$$

implies that

$$\omega^*(x_2) = R_2(x_0) \quad \forall x_2 \in X, \tag{5}$$

is player $2'$s equilibrium strategy, whereas

(ii)

$$\Pr\{X_1 = i \mid x_2 = i, x_0 = k\} \geq 1 - \delta \quad \forall i \in X. \tag{6}$$

implies that

$$\omega^*(x_2) = R_2(x_2) \quad \forall x_2 \in X, \tag{7}$$

is player $2'$s equilibrium strategy.

Consider the limit where both kinds of imperfections vanish. Let P^{mn} be a stochastic matrix defined on the state space X^2 with components $p^{mn}_{ij} > 0$ for all for all $(i,j) \in X^2$, and $\sum_i p^{mn}_{ij} = 1$. Let I denote the identity matrix on the space X^2 with components p^0_{ij}, where $p^0_{ii} = 1$ for all i and $p^0_{ij} = 0$ for all $i \neq j$. Then, imperfections tend to vanish if both P^{10} and P^{21} converge to I.

Definition 1 *P^{10} converges faster to I than P^{21} if*

$$\lim_{P^{10},P^{21}\to I} \left|\left|\frac{P^{10} - I}{P^{21} - I}\right|\right| = 0, \tag{8}$$

and P^{21} converges faster to I than P^{10} if

$$\lim_{P^{10},P^{21}\to I} \left|\left|\frac{P^{21} - I}{P^{10} - I}\right|\right| = 0, \tag{9}$$

where

$$\left|\left|\frac{P^{mn} - I}{P^{qr} - I}\right|\right| := \max_{(i,j,k,l)} \left\{ \left|\frac{p^{mn}_{ij} - p^0_{ij}}{p^{qr}_{kl} - p^0_{kl}}\right| \right\}. \tag{10}$$

For instance if distortions are uniform with $p^{10}_{ij} = \theta$ and $p^{21}_{ij} = \psi$ for all $i \neq j$. Then, $\left|\left|\frac{P^{10}-I}{P^{21}-I}\right|\right| = \frac{(N-1)\theta}{\psi}$ where N denotes the number of elements of the set X. In this case P^{10} converges faster to I as P^{21} if $\lim_{\theta,\psi\to 0} \frac{\theta}{\psi} = 0$ whereas P^{21} converges faster as P^{10} if $\lim_{\theta,\psi\to 0} \frac{\psi}{\theta} = 0$.

Lemma 2 **(i)** *Suppose* P^{10} *converges faster to* I *as* P^{21}*, then*

$$\lim_{P^{10},P^{21}\to I} \Pr(X_1 = k \mid x_2 = i, x_0 = k) = 1 \quad \forall (i,k) \in X^2. \tag{11}$$

(ii) *Suppose* P^{21} *converges faster to* I *as* P^{10}*, then*

$$\lim_{P^{10},P^{21}\to I} \Pr(X_1 = i \mid x_2 = i, x_0 = k) = 1 \quad \forall\, (i,k) \in X^2. \tag{12}$$

Proof By (2)–(3) one obtains:

$$\begin{aligned} \Pr\{X_1 = k \mid x_2 = i, x_0 = k\} &= \frac{p^{21}_{ik} p^{10}_{kk}}{\sum_j p^{21}_{ij} p^{10}_{jk}} \\ &= \frac{1}{1 + \sum_{j\neq k} \frac{p^{10}_{jk} p^{21}_{ij}}{p^{21}_{ik} p^{10}_{kk}}}, \end{aligned} \tag{13}$$

$$\begin{aligned} \Pr\{X_1 = i \mid x_2 = i, x_0 = k\} &= \frac{p^{21}_{ii} p^{10}_{ik}}{\sum_j p^{21}_{ij} p^{10}_{jk}} \\ &= \frac{1}{1 + \sum_{j\neq k} \frac{p^{21}_{ij} p^{10}_{jk}}{p^{10}_{ik} p^{21}_{ii}}}. \end{aligned} \tag{14}$$

(i) Condition (8) implies

$$\lim_{P^{10},P^{21}\to I} \frac{p^{10}_{jk}}{p^{21}_{ik}} = 0, \quad \forall j \neq k. \tag{15}$$

Also notice that $p^{21}_{ij} \in (0,1)$ and $\lim_{P^{10}\to I} p^{10}_{kk} = 1$. In conjunction with (15) one thus obtains

$$\lim_{P^{10},P^{12}\to I} \sum_{j\neq k} \frac{p^{10}_{jk} p^{21}_{ij}}{p^{21}_{ik} p^{10}_{kk}} = 0. \tag{16}$$

(15)–(16) yield (11).

(ii) Condition (9) implies

$$\lim_{P^{10},P^{21}\to I} \frac{p^{21}_{ij}}{p^{10}_{ik}} = 0 \quad \forall j \neq i. \tag{17}$$

The rest of the proof is similar to (i) and therefore omitted. ■

Just as in the example, we call the equilibrium outcomes of the associated *simultaneous–move* game without any imperfection "Cournot outcomes", and we call the subgame perfect equilibrium outcome of the *sequential–move* game without any imperfection the "Stackelberg outcome".[3]

Proposition 1 *The set of pure strategy equilibrium outcomes of the perturbed game converges*

(i) *to the set of "Cournot outcomes" if P^{10} converges faster to I as P^{21},*

(ii) *in probability to the "Stackelberg outcome" if P^{21} converges faster to I as P^{10}.*

Proof By the regularity assumption (1) there exists an $\epsilon_1 > 0$ such that

$$\max\{||P^{10} - I|| \; , \; ||P^{21} - I||\} \leq \epsilon_1$$

implies that player 1 responds to $2's$ equilibrium strategy $\omega^*(x_2)$ just as if there were no imperfections at all

$$x_0 = \arg\max_x u_1(x, \omega^*(x_2)) \quad \text{s.t.} \quad x = x_2. \tag{18}$$

(i) Suppose (8) holds. Then, by *Lemma 1(i)* and *Lemma 2(i)*, there exists an $\epsilon_2 > 0$ such that $\max\{||P^{10} - I|| \; , \; ||P^{21} - I||\} = ||P^{21} - I|| \leq \epsilon_2$ implies

$$\omega^*(x_2) = R_2(x_0) \quad \forall x_2 \in X. \tag{19}$$

Let $||P^{21} - I|| \leq \min\{\epsilon_1, \epsilon_2\}$. Then, (18) and (19) hold. Hence player $1's$ and $2's$ equilibrium action is

$$x_0 = R_1(y^*), \quad y^* = R_2(x_0), \tag{20}$$

and the set of equilibrium outcomes of the perturbed game coincides with the set of "Cournot outcomes".

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[3] By the regularity assumption (1), the sequential game without any imperfection has an unique subgame perfect equilibrium outcome.

(ii) Suppose (9) holds. Then, by *Lemma 1(ii)* and *Lemma 2(ii)*, there exists an $\epsilon_2 > 0$ such that $\max\{||P^{10} - I||\ ,\ ||P^{21} - I||\} = ||P^{10} - I|| \leq \epsilon_2$, implies that $2's$ equilibrium strategy is

$$\omega^*(x_2) = R_2(x_2) \quad \forall\, x_2 \in X. \tag{21}$$

Let $||P^{10} - I|| \leq \min\{\epsilon_1, \epsilon_2\}$. Then, (18) and (21) hold, and $1's$ and $2's$ equilibrium action is

$$x_0 = \arg\max_x u_1(x, R_2(x)), \quad y^* = R_2(x_2). \tag{22}$$

Finally, the probability that the observed signal x_2 coincides with $1's$ equilibrium action x_0 tends to one as imperfections tend to vanish. Hence, $\lim_{P^{10},P^{21}\to I} \Pr\{y^* = R_2(x_0)\} = 1$, and the equilibrium outcome of the perturbed game converges in probability to the "Stackelberg outcome". ■

By *Proposition 1(i)* the "Cournot outcome" is the unique pure strategy equilibrium outcome if there is only signal imperfection, $P^{10} = I$ as assumed in Bagwell (1995). However, by *Proposition 1(ii)* this outcome is generally not trembling hand perfect.

4 Many Imperfections

Denote player $1's$ equilibrium action by x_{T-t}, the payoff relevant variable by X_{T-1} and the signal by X_T. Then, a pure strategy for player 2 is a function ω from signals X_T to actions, so that the action $y \in Y$ taken when signal x_T is observed is $y = \omega(x_T)$. Further, in the associated sequential move game without any imperfections player $2's$ equilibrium strategy is $R_2(x_T)$, whereas it is $R_2(x_{T-t})$ in the associated simultaneous move game.[4]

Consider a Markov chain $x_{T-t} \to X_{T-t+1} \to \ldots \to X_{T-1} \to X_T$ with non stationary transition probabilities, and assume

Assumption 2 (Full Support) *Given a certain* $x_{n-1} = j$ *each* $x_n \in X$ *occurs with positive probability*

$$p_{ij}^{n,n-1} > 0 \quad \forall (i,j) \in X^2, \quad \forall\, n \geq 2. \tag{23}$$

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[4] R_2 denotes player $2's$ best response correspondence, i.e. $R_2(x)$ is the set of y values that maximize $u_2(x,y)$ over player $2's$ action set Y.

Assumption 3 (Small Error) *If $x_{n-t} = j$, then $x_n = j$ occurs with highest probability*

$$p_{jj}^{n,n-t} \geq p_{ij}^{n,n-t} \quad \forall i \neq j, \quad \forall t \in \{1, 2, ..., n\}. \tag{24}$$

Assumption 3 for instance holds if distortions are uniform and $p_{ij}^{n,n-1} = \theta_n$ for all $i \neq j$.

Lemma 3 *Consider the limit where imperfections vanish, $P^{n,n-1} \to I$ for all n.*

(i) *Suppose*

$$\lim_{P^{n,n-1} \to I \forall n} \left|\left| \frac{P^{T,T-1} - I}{P^{T-1,T-t} - I} \right|\right| = 0. \tag{25}$$

Then, player 2 behaves as the Stackelberg follower of the associated sequential move game without any imperfection, and

$$\omega^*(x_T) = R_2(x_T) \quad \forall \, x_T \in X. \tag{26}$$

(ii) *Suppose*

$$\lim_{P^{n,n-1} \to I \forall n} \left|\left| \frac{P^{T-1,T-t} - I}{P^{T,T-1} - I} \right|\right| = 0. \tag{27}$$

Then, player 2 behaves as the Cournot player of the associated simultaneous move game without any imperfection, and

$$\omega^*(x_T) = R_2(x_{T-t}) \quad \forall \, x_T \in X. \tag{28}$$

Proof **(i)** First notice that

$$\begin{aligned} \Pr\{X_{T-1} = i \mid x_T = i, x_{T-t} = k\} &= \frac{p_{ii}^{T,T-1} p_{ik}^{T-1,T-t}}{\sum_j p_{ij}^{T,T-1} p_{jk}^{T-1,T-t}} \\ &= \frac{1}{1 + \sum_{j \neq i} \frac{p_{ij}^{T,T-1} p_{jk}^{T-1,T-t}}{p_{ik}^{T-1,T-t} p_{ii}^{T,T-1}}}. \end{aligned} \tag{29}$$

Next, notice that

$$\lim_{P^{n,n-1}\to I,\,\forall n} \frac{p_{ij}^{T,T-1}}{p_{ik}^{T-1,T-t}} = 0$$

for all $j \neq i$, by (25), and that $p_{jk}^{T-1,T-t} \in (0,1)$ and $\lim_{P^{T,T-1}\to I} p_{ii}^{T,T-1} = 1$. Hence

$$\lim_{P^{n,n-1},\to I\,\forall n} \Pr\{X_{T-1} = i \mid x_T = i, x_{T-t} = k\} = 1 \quad \forall\,(i,k) \in X^2. \tag{30}$$

From this the assertion follows immediately by the regularity assumption (1).

(ii) Proceed as in (i). It turns out that (27) implies

$$\lim_{P^{n,n-1}\to I\,\forall n} \Pr\{X_{T-1} = k \mid x_T = i, x_{T-t} = k\} = 1 \quad \forall\,(i,k) \in X^2. \tag{31}$$

From this the assertion follows immediately by the regularity assumption (1).
■

Lemma 4 **(i)** $\left|\left|P^{T-1,T-t} - I\right|\right|$ *increases as the number of imperfections* t *increases; and*

(ii) *player* $1'$*s equilibrium action* x_{T-t} *becomes uninformative concerning the payoff relevant variable* X_{T-1}*, as the number* t *of imperfections increases.*

$$\lim_{t\to\infty} p_{ik}^{T-1,T-t} = \frac{1}{N} \quad \forall\,(i,k) \in X^2, \tag{32}$$

$$\left| p_{ik}^{T-1,T-t} - \frac{1}{N} \right| \leq (1 - Nb)^{t-1}. \tag{33}$$

There N *denotes the number of elements of the set* X*, and* $b := \min_n b^{n,n-1}$ *where* $b^{n,n-1} := \min_{i,j} p_{ij}^{n,n-1}$.

Proof **(i)** First notice that

$$\begin{aligned} p_{ij}^{n,m} &= \sum_z p_{iz}^{n,m+1} p_{zj}^{m+1,m} \\ &\geq \max_z \left\{ p_{iz}^{n,m+1} \right\} \sum_z p_{zj}^{m+1,m} \\ &= \max_z p_{iz}^{n,m+1}, \end{aligned} \tag{34}$$

and that

$$\max_z p_{iz}^{k,l} = p_{ii}^{k,l} \tag{35}$$

by *Assumption 3.* Therefore

$$\sum_{i\neq j} p_{ij}^{n,m} \geq \sum_{i\neq j} p_{ii}^{n,m+1} \geq \sum_{i\neq j} p_{ij}^{n,m+1}. \tag{36}$$

Finally notice that

$$||P^{k,l} - I|| = \max_j \left\{ \sum_{i\neq j} p_{ij}^{k,l} \right\}. \tag{37}$$

Hence, $||P^{n,m} - I|| \geq ||P^{n,m+1} - I||$ by (36) and (37), and the assertion follows immediately.

(ii) Define

$$B_{i\cdot}^{n,m} := \max_j p_{ij}^{n,m}, \quad b_{i\cdot}^{n,m} := \min_j p_{ij}^{n,m} \tag{38}$$

As shown in the *Appendix*

$$b_{i\cdot}^{n,n-1} \leq b_{i\cdot}^{n,n-2} \leq .... \leq b_{i\cdot}^{n,n-r} \leq B_{i\cdot}^{n,n-r} \leq ..... \leq B_{i\cdot}^{n,n-1}, \tag{39}$$

and

$$B_{i\cdot}^{n,n-r} - b_{i\cdot}^{n,n-r} \leq \left[B_{i\cdot}^{n,n-1} - b_{i\cdot}^{n,n-1}\right] [1 - Nb]^{r-1}. \tag{40}$$

By (39) $\lim_{t\to\infty} B_{i\cdot}^{T-1,T-t}$ and $\lim_{t\to\infty} b_{i\cdot}^{T-1,T-t}$ exist, and since $(1 - Nb) \in (0, 1)$, $\lim_{t\to\infty} B_{i\cdot}^{T-1,T-t} = \lim_{t\to\infty} b_{i\cdot}^{T-1,T-t}$ by (40). Therefore

$$\lim_{t\to\infty} \max_j p_{ij}^{T-1,T-t} = \lim_{t\to\infty} p_{ij}^{T-1,T-t} = \lim_{t\to\infty} \min_j p_{ij}^{T-1,T-t}, \tag{41}$$

and (32) follows immediately in conjunction with *Assumption 3.*

(33) holds since

$$\begin{aligned} \left| p_{ik}^{T-1,T-t} - \frac{1}{N} \right| &\leq B_{i\cdot}^{T-1,T-t} - b_{i\cdot}^{T-1,T-t} \\ &\leq \left[B_{i\cdot}^{T-1,T-2} - b_{i\cdot}^{T-1,T-2}\right] (1 - Nb)^{t-2}, \end{aligned} \tag{42}$$

by (39) and (40), and since $B_{i\cdot}^{T-1,T-2} \leq 1-(N-1)b$ and $b_{i\cdot}^{T-1,T-2} \geq b$. ■

Note that the "Cournot outcome" results only if

$$\lim_{P^{n,n-1}\to I\,\forall n} \big|\big|P^{T-1,T-t}-I\big|\big| = 0$$

, by *Lemma 3* and since $\lim_{P^{n,n-1}\to I\,\forall n} \big|\big|P^{T,T-1}-I\big|\big| = 0$. By *Lemma 4* $left|\big|P^{T-1,T-t}-I\big|$ increases as the number of imperfections t increases. Hence, the prerequisites for Bagwell's result become more and more restrictive as the number of imperfections increases. For an illustration consider the following *Example 1*.

Example 1 *Consider the simple* 2×2 *game of* Section 1 *with payoffs as stated in* Figure 1. *There Player* 2 *behaves as a Cournot player and* $\omega^*(x_T) = R_2(x_{T-t})$ *for all* x_T *if and only if*

$$\Pr\{X_{T-1}=k \mid x_T=j, x_{T-t}=k\} \geq 1/2 \quad \forall\,(j,k)\in\{C,S\}^2. \tag{43}$$

First consider two imperfections $(t=2)$ *and assume*

$$p_{i,j}^{T,T-1} = p_{i,j}^{T-1,T-2} = \theta < 1/2 \quad \forall\, i\neq j. \tag{44}$$

Then

$$\Pr\{X_{T-1}=k \mid x_T=k, x_{T-2}=k\} = \frac{(1-\theta)^2}{(1-\theta)^2+\theta^2} > \frac{1}{2}, \tag{45}$$

$$\Pr\{X_{T-1}=k \mid x_T=j, x_{T-2}=k\} = \frac{\theta(1-\theta)}{2\theta(1-\theta)} = \frac{1}{2} \tag{46}$$

for all $j\neq k$. *Hence, (43) holds, and there exists an equilibrium where* 2 *behaves as a Cournot player.*

Now introduce a third imperfection and assume

$$p_{i,j}^{T-2,T-3} = \phi < 1/2 \quad \forall\, i\neq j. \tag{47}$$

Then

$$\begin{aligned}\Pr\{X_{T-1}=k \mid x_T=j, x_{T-3}=k\} &= \frac{p_{jk}^{T,T-1}\,p_{kk}^{T-1,T-3}}{\sum_z p_{jz}^{T,T-1}\,p_{zk}^{T-1,T-3}} \\ &= \frac{\theta[(1-\theta)(1-\phi)+\theta\phi]}{\theta[(1-\theta)(1-\phi)+\theta\phi]+(1-\theta)[(\theta(1-\phi)+(1-\theta)\phi]} < \frac{1}{2}\end{aligned} \tag{48}$$

for all $j \neq k$*. Hence, (43) is violated, and there exists no equilibrium where* 2 *behaves as a Cournot player.*

Moreover $\Pr\{X_{T-1} = j \mid x_T = j, x_{T-2} = k\} \geq 1/2$ *for all* j, k*, in this case, and Player* 2 *behaves as a "Stackelberg follower".*

Finally, if the number of imperfections is sufficiently high and the signal imperfection sufficiently small, then players behave as in the associated sequential move game without signal imperfection where $p_{ii}^{T,T-1} = 1$ for all i.

Proposition 2 *There exists a real number* $m > 0$ *and an* $\epsilon > 0$ *such that* $t \geq m$ *and* $\left|\left|P^{T,T-1} - I\right|\right| \leq \epsilon$ *implies that player* 2 *and player* 1 *behave just as in the associated sequential move game without signal imperfection* ($P^{T,T-1} = I$)*, and*

$$\omega^*(x_T) = R_2(x_T) \quad \forall\, x_T \in X, \tag{49}$$

$$\begin{aligned} x_{T-t} &= \operatorname*{argmax}_x \mathrm{E}\left\{u_1(X_T, R_2(X_T) \mid x_{T-t} = x\right\} \\ &= \operatorname*{argmax}_x \sum_j u_1(X_T = j, R_2(X_T = j)) p_{jx}^{T-1,T-t}. \end{aligned} \tag{50}$$

Proof By the regularity assumption there exists a $\delta > 0$ such that

$$\Pr\left\{X_{t-1} = i \mid x_t = i, x_{T-t} = k\right\} \geq 1 - \delta \quad \forall\, j \in X \tag{51}$$

implies that (49) is $2's$ equilibrium strategy. And there exists an $\epsilon \in (0, \delta)$ such that

$$\left|\left|P^{T,T-1} - I\right|\right| \leq \epsilon \tag{52}$$

implies that player 1 responds to (49) with (50).

Choose

$$t \geq 1 + \frac{\ln \frac{\delta - \epsilon}{N(2-\epsilon-\delta)}}{\ln(1 - Nb)} =: m. \tag{53}$$

Then,

$$\begin{aligned} (1-\epsilon)\left[\frac{1}{N} - (1-Nb)^{t-1}\right] &\geq (1-\delta)\left[\frac{1}{N} + (1-Nb)^{t-1}\right] \\ &> (1-\delta)\left[\frac{1}{N} + (1-Nb)^{t}\right], \end{aligned} \tag{54}$$

which in turn implies that

$$(1-\epsilon)\frac{\frac{1}{N}-(1-Nb)^{t-1}}{\frac{1}{N}+(1-Nb)^{t}} > 1-\delta. \tag{55}$$

Let (52) hold. Then $p_{ii}^{T,T-1} \leq 1-\epsilon$, and in conjunction with (33) one obtains

$$\begin{aligned}(1-\epsilon)\frac{\frac{1}{N}-(1-Nb)^{t-1}}{\frac{1}{N}+(1-Nb)^{t}} &\leq p_{ii}^{T,T-1}\frac{p_{ik}^{T-1,T-t}}{p_{ik}^{T,T-t}} \\ &= \Pr\{X_{t-1}=i \mid x_t=i, x_{T-t}=k\}. \end{aligned}\tag{56}$$

And (51) holds by (55)–(56). ■

Appendix

Here we complete the proof of *Lemma 4(ii).*

(1) First we prove (39). Notice that for all $n > m+1$

$$\begin{aligned} b_{i\cdot}^{n,m} &= \min_j \left\{ \sum_z p_{iz}^{n,m+1} p_{zj}^{m+1,m} \right\} \\ &\geq \min_j \left\{ b_{i\cdot}^{n,m+1} \sum_z p_{zj}^{m+1,m} \right\} \\ &= b_{i\cdot}^{n,m+1}, \end{aligned}\tag{57}$$

and

$$\begin{aligned} B_{i\cdot}^{n,m} &= \max_j \left\{ \sum_z p_{iz}^{n,m+1} p_{zj}^{m+1,m} \right\} \\ &\leq \max_j \left\{ B_{i\cdot}^{n,m+1} \sum_z p_{zj}^{m+1,m} \right\} \\ &= B_{i\cdot}^{n,m+1}. \end{aligned}\tag{58}$$

Choose $m = n-t$, $n-t+1, ..., n-2, n-1$, then 39 follows immediately.

(2) Second we prove condition (40). Notice that

$$\begin{aligned}
B_{i\cdot}^{n,m} - b_{i\cdot}^{n,m} &:= \max_j \left\{ \sum_z p_{iz}^{n,m+1} p_{zj}^{m+1,m} \right\} - \min_j \left\{ \sum_z p_{iz}^{n,m+1} p_{zj}^{m+1,m} \right\} \\
&= \max_{j,k} \left\{ \sum_z p_{iz}^{n,m+1} [p_{zj}^{m+1,m} - p_{zk}^{m+1,m}] \right\} \\
&\leq \max_{j,k} \left\{ B_{i\cdot}^{n,m+1} \sum_z^+ [p_{zj}^{m+1,m} - p_{zk}^{m+1,m}] + b_{i\cdot}^{n,m+1} \sum_z^- [p_{zj}^{m+1,m} - p_{zk}^{m+1,m}] \right\} \\
&= \max_{j,k} \left\{ [B_{i\cdot}^{n,m+1} - b_{i\cdot}^{n,m+1}] \sum_z^+ [p_{zj}^{m+1,m} - p_{zk}^{m+1,m}] \right\}. \qquad (59)
\end{aligned}$$

There, $\sum_z^+$ denotes the sum over the elements with $p_{zj}^{m+1,m} \geq p_{zk}^{m+1,m}$, and $\sum_z^-$ denotes the sum over the elements with $p_{zj}^{m+1,m} < p_{zk}^{m+1,m}$ for a fixed j and k.

Let $\omega(j,k)$ denote the number of $z's$ with $p_{zj}^{m+1,m} \geq p_{zk}^{m+1,m}$ for a given j and k. Then

$$\sum_z^+ p_{zk}^{m+1,m} \geq \sum_z^+ b \;=\; b\omega(j,k), \qquad (60)$$

and

$$\sum_z^+ p_{zj}^{m+1,m} = 1 - \sum_z^- p_{zj}^{m+1,m} \leq 1 - \sum_z^- b \;=\; 1 - b(N - \omega(j,k)). \qquad (61)$$

(60)–(61) yield

$$\sum_z^+ (p_{zj}^{m+1,m} - p_{zk}^{m+1,m}) \leq 1 - Nb \quad \forall\, j,k, \qquad (62)$$

and in conjunction with (59) one obtains

$$B_{i\cdot}^{n,m} - b_{i\cdot}^{n,m} \leq \left[B_{i\cdot}^{n,m+1} - b_{i\cdot}^{n,m+1} \right] [1 - Nb]. \qquad (63)$$

Succesively choose $m = n-r, n-r+1, ..., n-2$, then (40) follows immediately.

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