

# The Term Structure of Interest Rates when the Growth Rate is Unobservable

Frank Riedel\*

Graduate College Applied Microeconomics  
Humboldt University  
riedel@wiwi.hu-berlin.de

September 4, 1997

## Abstract

The effect of incomplete information on the term structure of interest rates is examined in the framework of a pure exchange economy under uncertainty. When the growth rate of the aggregate endowment is known, the term structure is flat and deterministic. When agents do not observe the (constant) growth rate, interest rates are stochastic and the term structure turns out to be linearly decreasing. As a possible explanation of this astonishing fact, we suggest that rational and risk-averse investors consider long-term bonds as a good hedge against unfavorable realizations of the growth rate.

*JEL Classification No.: D5, D9, E4, G1*

*Keywords : Term Structure of Interest Rates, Incomplete Information*

## Introduction

What happens to the term structure of interest rates when some factors are unobservable? We study this question using a pure exchange economy in which aggregate endowment is modeled as a diffusion whose drift coefficient (the expected growth rate) is unobservable, but fixed over time. Rational

---

\*The author is grateful to Michael Kaul for a helpful discussion and to Hans Föllmer for his valuable advice. Support from the Deutsche Forschungsgemeinschaft, Graduiertenkolleg Angewandte Mikroökonomik is gratefully acknowledged.

and risk-averse agents observe only the output data and engage in Bayesian inference about the growth rate. The result is that the term structure is flat under perfect information but *linearly decreasing* with incomplete information. The loss of information lowers the yields of long-term bonds. At first sight, this seems to be a puzzle for the following reasons. With complete information the short rate is known and constant for all times. Under incomplete information the short rate becomes a stochastic process; therefore, agents are running a risk if they buy long-term bonds. Because of the usual risk-aversion argument one would expect that agents would require a positive term premium in equilibrium and that the term structure would be rising. But the converse is the case in our model. A possible explanation may be that rational and risk-averse investors consider long-term bonds as a good hedge against unfavorable realizations of the growth rate.

The effect of incomplete information on interest rates has also been investigated by Detemple [2], Dothan and Feldman [3], and Feldman [5] in a series of papers. The work of Dothan and Feldman [3] and Detemple [2] concentrates on methodological issues. They characterize equilibrium asset prices using partial differential equations and describe the filtering of the unobservable factor.

An explicit investigation of the resulting term structure can be found in Feldman [5]. He uses a Cox-Ingersoll-Ross [1] type production economy where the expected growth rate of the physical asset is unobservable. The growth rate is modelled as a stationary Ornstein-Uhlenbeck-process.

With complete information, the resulting short rate is also an Ornstein-Uhlenbeck process like in Vasiček's [10] model. In particular, interest rates are already stochastic with complete information. Under incomplete information, the short rate exhibits still the mean-reverting property of the Ornstein-Uhlenbeck process. The difference is that, due to the filtering error, the diffusion coefficient of the short rate changes over time. The resulting term structure belongs to the class of Hull-White [6] models.

A drawback of Feldman's model seems to be that the price process of the physical asset and the dynamics of the unobservable growth rate are driven by the same one-dimensional Brownian motion. This leads to a high correlation of observable and unobservable data. In particular, the additional term in the diffusion coefficient that stems from the filtering error decreases exponentially to 0.

Feldman shows that there may be an additional hump in the yield curve caused by the additional term in the diffusion coefficient. However, the effect on the term structure is less dramatic than in our model.

In the model we present, the unobservable factor is independent of the Brownian motion that drives the dynamics of the aggregate endowment.

Moreover, with complete information interest rates are deterministic whereas they are stochastic when the growth rate is unobservable.

The loss of information measured by the increase in variance of the short rate is larger in our model than in Feldman's. The diffusion coefficient that enters the model decreases also to 0 in the long run, but only at the order of  $\frac{1}{x}$ , not exponentially.

The paper is organized as follows. In the next section the setup of the economy is described. In section 2 the Bayesian estimate of the unobservable growth rate is calculated. The last section presents the results on the term structure and offers a heuristic explanation of the puzzling result.

## 1 The economy

Our aim is to understand the effect of incomplete information on interest rates. We introduce the standard exchange model of financial economics in continuous time. A single, perishable good is traded in the economy whose aggregate output evolves according to the stochastic differential equation

$$\frac{dC}{C} = \mu dt + dW \quad , C_0 = 1, \quad (1)$$

where  $W$  is a one-dimensional Brownian motion defined on a complete probability space  $(\Omega, \mathcal{F}, P)$ . The additional feature here is that the growth rate  $\mu$  is not constant, but a random variable independent of the Brownian motion  $W$ . In the case of incomplete information,  $\mu$  will not be observed by the agents. Instead, they have a prior belief of the distribution of the growth rate that they update according to the observation of the realized output  $C$ . For simplicity,  $\mu$  is taken to be normally distributed,  $\mu \sim N(m, 1)$ . In the case of complete information, agents observe the realization of the growth rate  $\mu$  at time 0. Their information at time  $t$  is given by the filtration

$$\mathcal{F}_t = \sigma(\mu, (C_s)_{s \leq t}). \quad (2)$$

When agents observe realized outputs only their information is given by the filtration

$$\mathcal{G}_t = \sigma((C_s)_{s \leq t}). \quad (3)$$

Let  $T > 0$  be the fixed horizon of the economy.

The consumption space  $L^c$  resp.  $L^i$  is the space of nonnegative, square integrable processes that are adapted and progressively measurable with respect to  $\mathcal{F}$  in the case of complete information resp. with respect to  $\mathcal{G}$  in the

case of incomplete information:

$$L^c = \{C \in \mathcal{L}^2(\Omega, \mathcal{O}(\mathcal{F}), P) : C \geq 0\} \quad (4)$$

$$L^i = \{C \in \mathcal{L}^2(\Omega, \mathcal{O}(\mathcal{G}), P) : C \geq 0\} \quad (5)$$

where  $\mathcal{O}(\mathcal{F})$  denotes the  $\sigma$ -field of  $\mathcal{F}$ -progressively measurable,  $\mathcal{F}$ -adapted events. A representative agent has preferences over consumption streams given by a time-additive von Neumann-Morgenstern expected utility functional  $U$ :

$$U(D) = E \int_0^T e^{-\rho t} u(D_t) dt. \quad (6)$$

The felicity function  $u$  exhibits constant relative risk aversion  $\delta$ , that is

$$u(x) = \frac{x^{1-\delta}}{1-\delta} \quad (7)$$

for some  $\delta > 0$ . The case  $\delta = 1$  is allowed for and corresponds to log-utility.

Two securities are traded. The first one is referred to as the "stock" with (ex-dividend) price  $S$  and yields a continuous dividend of  $C_t dt$ , that is, it can be interpreted as the "market portfolio" promising to pay out the market's dividend in every point of time  $t$ . The second one is the locally riskless money account  $\beta$  that evolves according to

$$\frac{d\beta}{\beta} = r_t dt \quad (8)$$

where  $r$  denotes the short rate of interest at which instantaneous borrowing or lending is possible. Both price processes  $S$  and  $\beta$  (and therefore  $r$ ) are determined endogeneously in equilibrium.

The representative agent initially owns one share of the stock.

The asset prices  $S$  and  $\beta$  are adapted to the filtration at hand, that is, to  $\mathcal{F}$  in the case of complete information and to  $\mathcal{G}$  in the case of incomplete information. Furthermore,  $S$  is a semimartingale.

The agent takes these prices as given and chooses a portfolio/consumption strategy  $((\theta, \eta), D)$  that is  $\mathcal{F}$ - resp.  $\mathcal{G}$ -adapted and satisfies the integrability conditions

$$D \in L^c \text{ resp. } L^i \quad (9)$$

$$E \int_0^T \theta_u^2 d[S]_t < \infty \quad (10)$$

$$E \int_0^T \eta_u d\beta_u < \infty. \quad (11)$$

The portfolio strategy  $(\theta, \eta)$  in the stock and the money account finances the consumption stream  $D$ , that is, the value of the portfolio  $V_t = \theta_t S_t + \eta_t \beta_t$  meets the conditions

$$V_0 = S_0 \quad (12)$$

$$dV = \theta(dS + Cdt) + \eta dt - Ddt \quad (13)$$

$$V_T \geq 0. \quad (14)$$

A strategy  $((\theta, \eta), D)$  is called admissible when it meets the conditions (9) through (14).

**Definition 1.1** *A representative agent equilibrium consists of a pair of  $\mathcal{F}$  resp.  $\mathcal{G}$ -adapted asset prices  $(S, \beta)$  such that it is rational for the representative agent to consume aggregate output:*

$$C = \arg \max_{D \text{ admissible}} U(D). \quad (15)$$

## 2 Estimating the growth rate

Before deriving the equilibrium and the term structure it is useful to introduce the *innovation process*  $\bar{W}$  for the case of incomplete information. This is a Brownian motion which is *observable*, i.e.  $\mathcal{G}$ -adapted, and which carries the same information as the observed consumption data

$$\mathcal{F}^{\bar{W}} = \mathcal{G}. \quad (16)$$

When agents do not observe the expected growth rate  $\mu$  they estimate it on the basis of their observations. Set

$$\hat{\mu}_t = E[\mu | \mathcal{G}_t]. \quad (17)$$

Note that the process

$$\epsilon_t = \mu t + W_t \quad (18)$$

is observable because of

$$\epsilon_t = \log C_t + \frac{1}{2}t. \quad (19)$$

**Theorem 2.1** *The process*

$$\bar{W}_t = \epsilon_t - \int_0^t \hat{\mu}_u du \quad (20)$$

is a  $\mathcal{G}$ -adapted Brownian motion that carries the same information as the consumption data do:

$$\mathcal{G}_t = \sigma((\bar{W}_u)_{u \leq t}). \quad (21)$$

The best estimate of the expected growth rate  $\mu$  is

$$\hat{\mu}_t = \frac{\epsilon_t + m}{1 + t} \quad (22)$$

$$= m + \int_0^t \frac{1}{1 + u} d\bar{W}_u. \quad (23)$$

In terms of the innovation process  $\bar{W}$  the dynamics of aggregate consumption are

$$\frac{dC_t}{C_t} = \hat{\mu}_t dt + d\bar{W}_t. \quad (24)$$

PROOF : This is a simple corollary to the main theorem of filtering theory (cf. Liptser/Shiryayev [8]). Since everything is very simple here, we sketch the proof. By its definition (20)  $\bar{W}$  is a continuous, observable process, because  $\epsilon$  is continuous and observable. The following calculation shows that it is a martingale

$$E [\bar{W}_{t+s} - \bar{W}_t | \mathcal{G}_t] = E [W_{t+s} - W_t | \mathcal{G}_t] + \int_t^{t+s} E [\mu - \hat{\mu}_u | \mathcal{G}_t] du = 0$$

where one uses the fact that the increment of  $W$  is independent of its past and Fubini's theorem for conditional expectations. The quadratic variation is

$$[\bar{W}]_t = [\epsilon]_t = t.$$

By Levy's theorem,  $\bar{W}$  is a Brownian motion.

Calculating the estimate  $\hat{\mu}$  is a standard exercise in Bayesian inference:

$$\hat{\mu}_t = \frac{m + \epsilon_t}{1 + t}.$$

Apply Itô's formula (and neglect the drift term because  $\hat{\mu}$  is a martingale!) to obtain the dynamics

$$d\hat{\mu}_t = \frac{1}{1 + t} d\bar{W}_t.$$

Since  $\hat{\mu}$  can be written as the sum of a stochastic integral over  $\bar{W}$  with a deterministic integrand and a constant, it is clear that  $\hat{\mu}$  is  $\mathcal{F}^{\bar{W}}$ -adapted. By (22)  $\epsilon_t = (1+t)\hat{\mu}_t - m$  is  $\mathcal{F}^{\bar{W}}$ -adapted and also  $C_t = \exp(\epsilon_t + \frac{1}{2}t)$ , hence

$$\mathcal{F}^{\bar{W}} = \mathcal{G}.$$

□

For future use, we state the following

**Corollary 2.1** *The conditional distribution of the increment  $\epsilon_T - \epsilon_t$  given the past  $\mathcal{F}$  resp.  $\mathcal{G}$  is*

$$\mathcal{L}(\epsilon_T - \epsilon_t | \mathcal{F}_t) = N(\mu(T-t), (T-t)) \quad (25)$$

$$\mathcal{L}(\epsilon_T - \epsilon_t | \mathcal{G}_t) = N\left(\hat{\mu}(T-t), (T-t)\frac{1+T}{1+t}\right). \quad (26)$$

**PROOF :** In the case of complete information,  $\mu$  is known. The distribution of  $\epsilon_T - \epsilon_t = \mu(T-t) + W_T - W_t$  is determined by the increment of the Brownian motion  $W$  that is independent of the past.

Under incomplete information we use the representation (23) of the preceding theorem 2.1.

$$\begin{aligned} \epsilon_T - \epsilon_t &= \bar{W}_T - \bar{W}_t + \int_t^T \hat{\mu}_u du \\ &= \bar{W}_T - \bar{W}_t + \int_t^T (\hat{\mu}_u - \hat{\mu}_t) du + (T-t)\hat{\mu}_t \\ &= \bar{W}_T - \bar{W}_t + \int_t^T \int_t^u \frac{1}{1+s} d\bar{W}_s du + (T-t)\hat{\mu}_t \end{aligned}$$

An application of Fubini's theorem for stochastic integrals (cf. Protter [9], p. 160) gives

$$\begin{aligned} \epsilon_T - \epsilon_t &= \bar{W}_T - \bar{W}_t + \int_t^T \int_s^T \frac{1}{1+s} du d\bar{W}_s + (T-t)\hat{\mu}_t \\ &= \int_t^T d\bar{W}_s + \int_t^T \frac{T-s}{1+s} d\bar{W}_s + (T-t)\hat{\mu}_t \\ &= \int_t^T \frac{1+T}{1+s} d\bar{W}_s + (T-t)\hat{\mu}_t \end{aligned}$$

The last term  $(T-t)\hat{\mu}_t$  is known at time  $t$ . The first term  $\int_t^T \frac{1+T}{1+s} d\bar{W}_s$  is the increment of a stochastic integral over the Brownian motion  $\bar{W}$  with a deterministic integrand. It is therefore normally distributed and its variance

is

$$\begin{aligned}
E \left[ \left( \int_t^T \frac{1+T}{1+s} d\bar{W}_s \right)^2 \middle| \mathcal{F}_t \right] &= \int_t^T \frac{(1+T)^2}{(1+s)^2} ds \\
&= (1+T)^2 \left( \frac{1}{1+t} - \frac{1}{1+T} \right) \\
&= \frac{(1+T)(T-t)}{1+t}.
\end{aligned}$$

□

In the next section, it will turn out that the uncertainty in the economy is essentially contained in the increment of the Brownian motion with drift  $\epsilon$ . The variance of this increment increases linearly with time in the case of known drift  $\mu$ . But note the big loss of information when one has to estimate the drift! The conditional variance of the increment  $\epsilon_T - \epsilon_t$  becomes a *quadratic* function of time. The astonishing consequences of this fact will be the subject of the next section.

### 3 Equilibrium and the Term Structure

We are now able to derive the equilibrium of the market and the corresponding interest rates. We proceed along the lines of Duffie-Huang [4]. An Arrow-Debreu state-price process is given by the marginal utility of the representative agent. Set

$$\psi_t = e^{-\rho t} u'(C_t). \quad (27)$$

We use  $\psi$  as a discount factor for the asset prices. The stock price  $S$  is chosen such that the discounted value is equal to the discounted expected value of the future dividend stream:

$$S_t \psi_t = E \left[ \int_t^T C_u \psi_u du \middle| \mathcal{F}_t \right]. \quad (28)$$

Under incomplete information, we use the same (observable!) state-price process  $\psi$  but the smaller filtration  $\mathcal{G}$ :

$$\hat{S}_t \psi_t = E \left[ \int_t^T C_u \psi_u du \middle| \mathcal{G}_t \right]. \quad (29)$$

In the same manner, the short rate  $r$  resp.  $\hat{r}$  is determined to make a martingale of the discounted money account:

$$\beta_t \psi_t = E [\beta_T \psi_T | \mathcal{F}_t] \quad (30)$$

$$\hat{\beta}_t \psi_t = E [\beta_T \psi_T | \mathcal{G}_t]. \quad (31)$$

We obtain the following theorem:

**Theorem 3.1** *Under complete information, the representative agent equilibrium is given by*

$$S_t = C_t \frac{1 - \exp(-\gamma(T-t))}{\gamma} \quad (32)$$

$$r_t = \rho + \delta\mu - \frac{(1+\delta)\delta}{2} \quad (33)$$

where  $\gamma = \rho - (1-\delta)(\mu + \delta)$ . In the case of incomplete information, the equilibrium is

$$\hat{S}_t = C_t F \left( T-t, -\rho + (1-\delta)(\hat{\mu}_t - \frac{1}{2}) + \frac{(1-\delta)^2}{2}, \frac{(1-\delta)^2}{2(1+t)} \right) \quad (34)$$

$$\hat{r}_t = \rho + \delta\hat{\mu}_t - \frac{(1+\delta)\delta}{2} \quad (35)$$

with

$$F(\tau, a, b) = \int_0^\tau \exp(ax + bx^2) dx.$$

In both cases, the asset market spanned by the stock and the money market account is complete.

The proof is given in the appendix.

Before we have a closer look at the resulting term structure, let us digress for a moment to reflect about a very interesting feature of the asset market under incomplete information. The asset market is complete although it is possible to define  $\mathcal{F}$ -adapted consumption streams that cannot be financed by trading in the stock and the money account. Or, in the language of derivative pricing, there are  $\mathcal{F}_T$ -measurable contingent claims that cannot be hedged, e. g.  $\mu$ . In this sense, one could argue that the asset market is incomplete.

However, incomplete information requires the admissible consumption streams or contingent claims to be  $\mathcal{G}$ -adapted. The filtration  $\mathcal{G}$  is generated by the observable Brownian motion  $\bar{W}$ . Since  $\bar{W}$  generates all  $\mathcal{G}$ -martingales, the asset market defined in terms of  $\mathcal{G}$  is complete. In a setting of mere utility maximization, the fact that incomplete information may lead to a complete asset market has been pointed out by Karatzas and Xue [7].

Now let us study the term structure. As far as the short rate is concerned, theorem 3.1 seems to be a very natural result: the short rate is constant under complete information whereas it is the best estimate of the former under incomplete information:

$$\hat{r}_t = E[r_t | \mathcal{G}_t]. \quad (36)$$

As a conditional expectation,  $\hat{r}$  is a martingale. It converges towards the unobservable value  $r$  of the full information world.

The puzzling effect appears when we look at the term structure. Under full information, the interest rate is known for all times at time 0 and therefore the term structure is flat,

$$y_t^T = r_0 = \delta\mu - \frac{(1+\delta)\delta}{2} \quad (37)$$

where  $y_t^T$  denotes the yield to maturity  $T$  at time  $t$ . The following proposition shows that the term structure is linearly decreasing under incomplete information!

**Theorem 3.2** *Under incomplete information, the yields to maturity  $\widehat{y}_t^T$  are*

$$\widehat{y}_t^T = \hat{r}_t - \frac{\delta^2 T - t}{2(1+t)} \quad (38)$$

PROOF : Let  $\widehat{B}_t^T$  be the price of a zero-coupon bond with maturity  $T$  at time  $t$ . Since  $\psi$  is a state-price process, we have

$$\widehat{B}_t^T \psi_t = E[\psi_T | \mathcal{G}_t].$$

That is

$$\widehat{B}_t^T = E\left[\exp\left(-\rho(T-t) - \delta\left(\epsilon_T - \epsilon_t - \frac{1}{2}(T-t)\right)\right) | \mathcal{G}_t\right].$$

We have to calculate the conditional Laplace transform of a normally distributed random variable (corollary 2.1)

$$\widehat{B}_t^T = \exp\left(-\rho(T-t) - \delta\hat{\mu}_t(T-t) + \frac{\delta^2}{2}(T-t)\frac{1+T}{1+t} - \frac{\delta}{2}(T-t)\right).$$

One obtains the yields

$$\begin{aligned} \widehat{y}_t^T &= -\frac{1}{T-t} \log \widehat{B}_t^T \\ &= \rho - \frac{\delta}{2} + \delta\hat{\mu}_t - \frac{\delta^2}{2} \frac{1+T}{1+t} \\ &= \hat{r}_t - \frac{\delta^2 T - t}{2(1+t)}. \end{aligned}$$

□

A linearly decreasing term structure is quite unexpected for two reasons: on the one hand, we are rather used to see increasing yield curves than decreasing ones; on the other hand, the decreasing yield curve seems to contradict the classical risk premium argument that goes as follows. When interest rates become stochastic as it is the case under incomplete information, *long term bonds become risky*. The traditional theory predicts that risk averse investors require a positive risk premium in order to buy such an asset. Since the final value of a bond is fixed, a positive risk premium would lower the bond's price and increase its yield. An increasing term structure would be the consequence!

In our case, we do have stochastic interest rates and risk averse investors, but a decreasing term structure, that is a *negative* risk premium. This result seems to be quite at odds with the classical risk premium argument.

However, we feel that the risk premium argument is quite plausible in general. The mistake lies, in our eyes, in the seemingly innocuous phrase "*long term bonds become risky*".

It is true that the value of the bond is fluctuating randomly over time under stochastic interest rates, but in contrast to risky assets such as stocks the final value of a zero-coupon-bond is fixed to be 1\$. In the eyes of a risk averse investor, it may be much riskier to roll over the capital at random short rates than to buy a long-term bond. Especially here, where the growth rate  $\mu$  of the economy is unknown, a long-term bond is a good hedge against possible low realizations of  $\mu$ . Therefore, the demand for long-term bonds increases. We emphasize that this explanation is only heuristic since we have only derived a no-trade equilibrium where we cannot *show* that the demand for long-term bonds is higher.

Note that the slope  $-\frac{\delta^2}{2(1+t)}$  of the term structure tends to 0 as time passes. This is natural because investors learn about  $\mu$  and the need to hedge against unfavorable realizations of the growth rate decreases.

## 4 Conclusion

We studied a pure exchange economy with a fixed rate of output growth  $\mu$ . When  $\mu$  is known, the resulting term structure of interest rates is flat and all interest rates are known - we are in a deterministic model.

When the growth rate  $\mu$  of the economy is unobservable, rational Bayesian agents estimate it on the basis of observed output data. The resulting yield curve is linearly decreasing. The heuristic explanation we offer for this unexpected effect is that long term bonds are a good hedge against unfavorable realizations of the growth rate. Rational and risk averse investors prefer in

this case the certainty of a future payment of 1\$ promised by the long term bond to the risk of investing in stocks or rolling over the money at a random short rate.

This result sheds new light on the concept of riskless assets, too.

When agents do not know whether the economy's output will eventually go down or not, they accept lower yields on long term bonds in exchange to the safety of the terminal payment the bonds offer. Therefore, although bond prices are fluctuating randomly, one should not consider them as risky assets like stocks in this context. Bonds are, by definition, *riskless at maturity*. Due to this property they are a good hedge against low future outputs.

## A Appendix: Proof of the equilibrium relations

The derivation of the equilibrium is given for the case of incomplete information only. Under complete information, the proof runs along the same lines and the calculations are easier.

Set  $\psi_t = e^{-\rho t} C_t^{-\delta}$ . By Itô's formula and (24)

$$d\psi_t = -\psi_t \left( \left( \rho + \delta \hat{\mu}_t - \frac{\delta(1+\delta)}{2} \right) dt + \delta d\bar{W}_t \right).$$

Set

$$\hat{r}_t = \rho + \delta \hat{\mu}_t - \frac{\delta(1+\delta)}{2} \tag{39}$$

and define  $\hat{\beta}$  via

$$d\hat{\beta} = \hat{r}_t \hat{\beta}_t dt, \quad \beta_0 = 1. \tag{40}$$

The candidate for the equilibrium stock price is

$$\hat{S}_t := \frac{1}{\psi_t} E \left[ \int_t^T C_u \psi_u du \mid \mathcal{G}_t \right]. \tag{41}$$

### Lemma A.1

$$\hat{S}_t = C_t F \left( T-t, -\rho + (1-\delta) \left( \hat{\mu}_t - \frac{1}{2} \right) + \frac{(1-\delta)^2}{2}, \frac{(1-\delta)^2}{2(1+t)} \right) \tag{42}$$

with

$$F(\tau, a, b) = \int_0^\tau \exp(ax + bx^2) dx.$$

PROOF :

$$\begin{aligned}
\hat{S}_t &= \frac{1}{\psi_t} E \left[ \int_t^T C_u \psi_u du \mid \mathcal{G}_t \right] \\
&= e^{\rho t} C_t^\delta E \left[ \int_t^T e^{\rho u} C_u^{1-\delta} du \mid \mathcal{G}_t \right] \\
&= C_t \int_t^T e^{-\rho(u-t)} E \left[ \left( \frac{C_u}{C_t} \right)^{1-\delta} \mid \mathcal{G}_t \right] du \\
&= C_t \int_t^T e^{-\rho(u-t)} E \left[ \exp \left( (1-\delta)(\epsilon_u - \epsilon_t) - \frac{1-\delta}{2}(u-t) \right) \mid \mathcal{G}_t \right] du .
\end{aligned}$$

Using the conditional distribution of the increment  $\epsilon_u - \epsilon_t$  given the observed data  $\mathcal{G}_t$  obtained in Corollary 2.1 yields

$$\begin{aligned}
\hat{S}_t &= C_t \int_t^T \exp \left( ((1-\delta)(\hat{\mu}_t - \frac{1}{2}) - \rho)(u-t) + \frac{(1-\delta)^2}{2}(u-t) \frac{1+u}{1+t} \right) du \\
&= C_t \int_0^{T-t} \exp \left( ((1-\delta)(\hat{\mu}_t - \frac{1}{2}) - \rho)x + \frac{(1-\delta)^2}{2} x \frac{1+t+x}{1+t} \right) du
\end{aligned}$$

This is (34) resp. (42).  $\square$

Introduce the price functional for consumption streams in  $L^i$ :  $\Psi(D) = E \int_0^T D_u \psi_u du$ . We prove next the plausible result that the "price"  $\Psi(D)$  of every affordable consumption stream  $D$  is less than the initial endowment  $S_0$ .

**Lemma A.2** *For every admissible strategy  $((\theta, \eta), D)$  we have*

$$\Psi(D) \leq S_0 .$$

PROOF : This is the standard result that the sum of the deflated value of the portfolio and deflated cumulative consumption  $V_t \psi_t + \int_0^t D_u \psi_u du$  is a martingale.

To see this, introduce the martingales

$$\begin{aligned}
M_t &= E \left[ \int_0^T C_u \psi_u du \mid \mathcal{G}_t \right] \\
N_t &= E \left[ \hat{\beta}_T \psi_T \mid \mathcal{G}_t \right] .
\end{aligned}$$

By the definitions (39), (40), and (41) we have

$$\begin{aligned}
M_t &= \hat{S}_t \psi_t + \int_0^t C_u \psi_u du \\
N_t &= \hat{\beta}_t \psi_t .
\end{aligned}$$

Hence,

$$dM_t = \hat{S}_t d\psi_t + \psi_t dS_t + d[\hat{S}, \psi]_t + \psi_t C_t dt \quad (43)$$

$$dN = \hat{\beta}_t d\psi_t + \psi_t d\hat{\beta}_t. \quad (44)$$

Now, using the dynamics of  $V$  as defined in (13) we obtain the dynamics of the deflated value of the portfolio

$$\begin{aligned} d(V_t \psi_t) &= \psi_t dV_t + V_t d\psi_t + d[V, \psi]_t \\ &= \psi_t \theta_t d\hat{S}_t + \psi_t \theta_t C_t dt + \psi_t \eta_t d\hat{\beta}_t - \psi_t D_t dt + V_t d\psi_t + \theta d[S, \psi] \end{aligned} \quad (45)$$

On the other hand, (43) and (44) imply

$$\begin{aligned} \theta_t dM_t + \eta_t dN_t &= \theta_t \hat{S}_t d\psi_t + \theta_t \psi_t d\hat{S}_t + \theta_t d[\hat{S}, \psi]_t + \theta_t \psi_t C_t dt \\ &\quad + \eta_t \hat{\beta}_t d\psi_t + \eta_t \psi_t d\hat{\beta}_t \\ &= V_t d\psi_t + \theta_t \psi_t d\hat{S}_t + \theta_t d[\hat{S}, \psi]_t + \theta_t \psi_t C_t dt + \eta_t \psi_t d\hat{\beta}_t \end{aligned} \quad (46)$$

Comparing terms in (46) and (45) yields

$$d(V_t \psi_t) = \theta_t dM_t + \eta_t dN_t - D_t \psi_t dt.$$

Therefore,  $V_t \psi_t + \int_0^t D_u \psi_u du$  is a martingale. Because of the no terminal debt condition  $V_T \geq 0$

$$\Psi(D) = E \int_0^T D_u \psi_u du \leq E \int_0^T (D_u \psi_u du + V_T \psi_T) = V_0 \psi_0 = S_0.$$

□

A standard Lagrange argument yields the optimality of the strategy  $((1,0), C)$ . Let  $((\theta, \eta), D)$  be an admissible strategy. Then by the above lemma A.2

$$U(D) \leq U(D) + S_0 - \Psi(D) = S_0 + E \int_0^T e^{-\rho t} (u(D_t) - D_t \psi_t) dt.$$

The expression  $u(D) - D\psi_t$  is maximized when the first order condition

$$u'(D) = \psi_t = u'(C_t)$$

holds. Hence,  $C$  is optimal and  $(\hat{S}, \hat{\beta})$  a representative agent equilibrium.

## References

- [1] Cox, J., Ingersoll, J., Ross, S. (1985) An Intertemporal General Equilibrium Model of Asset Prices *Econometrica* **53** 363-384
- [2] Detemple, J. (1986) Asset Pricing in a Production Economy with Incomplete Information *Journal of Finance* **41** 383-391
- [3] Dothan, M. U. , Feldman, D. (1986) Equilibrium Interest Rates and Multiperiod Bonds in a Production-Exchange Economy *Journal of Finance* **41** 369-82
- [4] Duffie, D. and Huang, C.F. (1985) Implementing Arrow-Debreu Equilibria by Continuous Trading of few Long-Lived Securities *Econometrica* **53** 1337-56
- [5] Feldman, D. (1989) The Term Structure of Interest Rates in a Partially Observable Economy *Journal of Finance* **44** 789-812
- [6] Hull, J. and White, A. (1990) Pricing Interest Rate Derivative Securities *Review of Financial Studies* **3** 573-92
- [7] Karatzas, I. , Xue, X. (1991) A Note on Utility Maximization under Partial Observations *Math. Finance* **1** 57-70
- [8] Liptser, R. S. and Shiriyayev, A. N. (1977) *Statistics of Random Processes I* Springer-Verlag, New York
- [9] Protter, P. (1995) *Stochastic Integration and Differential Equations* Springer-Verlag, New York
- [10] Vasiček, O. (1977) An Equilibrium Characterization of the Term Structure *Journal of Financial Economics* **5** 177-88