Signaling equilibria in a multi-unit English clock auction[1]

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Abstract

The present paper analyzes a simple multi-unit English clock auction under incomplete information and decreasing marginal valuations. That game has a unique equilibrium where bidders play a pooling strategy that ends the auction immediately in its first round when the types of bidders are "not very" different. However, when bidders' types are "very" different, that low price equilibrium is preserved only for certain prior beliefs, whereas for other priors the price goes up to the competitive price with positive probability. Interestingly, if the price increment goes to zero, the game has no separating equilibrium.

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1 Introduction

In multi-unit auctions, strategic demand reduction is a critical issue. Typically, bidders can contribute to lower the equilibrium price by bidding less than their true demand. This may give rise to low prices.[1]

The auction format that seems to be particularly vulnerable to strategic demand reduction is the simultaneous, ascending-bid auction. Indeed, if bidders have complete information and declining marginal valuations, and the price increment is sufficiently small, that auction game has a unique equilibrium, in which the auction ends immediately at the minimum price.[2]

The present paper analyzes the English clock auction under incomplete information and declining marginal valuations. We look at a tractable specification, with two bidders, two types, and two units. We show that the auction ends immediately, if bidders' types are not very different or the prior probability of the high type is sufficiently large. However, otherwise the price goes up to the competitive level with positive probability.

2 Model

There are two bidders (A and B) who bid for two identical objects. Each bidder is either type h (high), which occurs with probability $\mu_0 \in (0,1)$, or l (low), represented by the decreasing marginal valuations for the first resp. second unit, $v_h(1) > v_h(2) > v_l(1) > v_l(2) = 0$. Bidders' type is their private information.

The auction is an open, ascending–bid clock auction. There, the price clock goes up by the fixed increment $\Delta > 0$, starting at price $p = 0$, until there is no excess demand. In each round bidders simultaneously submit a bid $\beta_i(p) \in \{0,1,2\}$, $i = A, B$, which states how many units they demand at the given price $p \in \{0, \Delta, 2\Delta, \dots\}$.[3] If the sum of bids

[1] This was observed, among others, by Cramton (2005), Ausubel and Cramton (1995), Menezes (1996) for dynamic auctions, Back and Zender (1993) and Wilson (1979) for sealed bid auctions, and Engelbrecht-Wiggans and Kahn (1998) and Brusco and Lopomo (2002) for multi-object auctions with non–decreasing marginal valuations.

[2] See Grimm, Riedel, and Wolfstetter (2003) for the case of two bidders and Riedel and Wolfstetter (2004) for the case of an arbitrary number of bidders and units. A similar uniqueness result is also in Ausubel and Schwartz (1999); however, by assuming that bidding follows a given order of moves, they analyzes a sequential bargaining problem, rather than a simultaneous auction.

[3] Bidding strategies may depend on the history of the game, of course. In order to keep the notation simple, we do not make this dependence explicit.

does not exceed the supply 2, the game ends and bidders pay the current price for each of the $\beta_i(t)$ objects they get.

Once a bidder has bid on $k < 2$ units, he cannot later bid on more than k units; therefore, the sequence of bids must be non-increasing (activity rule).

Two additional assumptions are made:

A1 Players never play weakly dominated strategies; specifically, player l demands at least one unit at all prices up to $p_l := \sup\{p \mid p \leq v_l(1)\}$.

A2 the price increment Δ is sufficiently small: $\Delta < p_l$.

In the following we describe the history of the game by the current state, $s = (i, j)$, and the current price p; thereby, $s = (i, j)$ means that, in the previous round, bidder A demanded i and B demanded j units. We encounter "pooling" , "separating", and "delayed separating" equilibria. The latter involve some rounds of delay, where both types of bidders demand two units until they switch to the separating equilibrium strategy.

3 Uniqueness of low price equilibrium when types are "not very" different

First, we consider the case, when the difference between $v_h(2)$ and $v_l(1)$ is "small".

PROPOSITION 1 *If $v_h(2) < 2p_l + \Delta$, the game has a unique equilibrium outcome: the auction ends immediately, at price $p = 0$, and each bidder is awarded one unit.*

PROOF The proof is in two steps. First, we consider the subgames defined by the price $p = \Delta$ combined with the either state $(2, 1)$ or $(1, 2)$, and show that demand reduction is the unique equilibrium. Second, we use this result to show that immediate demand reduction by both players is the unique equilibrium, for all prior beliefs μ_0.

1) Consider the subgame described by the price $p = \Delta$ and the state $(2, 1)$ (the other case, when the state is $(1, 2)$ is similar). If player A is type l, the assertion is obviously true, since he can never obtain more than one unit. If player A is type h, the only interesting alternative to immediate reduction is to maintain a demand for two units up to price

$p_l + \Delta$, and thereafter reduce demand if the player B has not dropped out of the auction (which occurs if player B is type h) and otherwise maintain a demand for two units. If A plays that alternative strategy, he obtains two units at the price $p_l + \Delta$ if B is type l and otherwise one unit at price $p_l + 2\Delta$; therefore, his payoff is equal to

$$\pi_A^2 = (1 - \mu)(v_h(1) + v_h(2) - 2(p_l + \Delta)) + \mu(v_h(1) - (p_l + 2\Delta)).$$

Whereas if he plays the candidate equilibrium strategy, he obtains one unit immediately, at price $p = \Delta$, and thus earns the payoff

$$\pi_A^1 = v_h(1) - \Delta.$$

Since $v_h(2) < 2p_l + \Delta$, one obtains:

$$\pi_A^1 - \pi_A^2 = (1 - \mu)\left((2p_l + \Delta) - v_h(2)\right) + \mu(p_l - \Delta) > 0,$$

which proves the assertion.

2) Now consider the first round of the auction, when the price is $p = 0$. For a player type l it is a weakly dominant strategy to demand only one unit, because that assures him that the auction either ends immediately at price 0 (if the rival also demands one unit) or one round later, at price $p = \Delta$ (if the rival demands two units), by 1), whereas if he demands two units, the auction cannot end at a price below $p = \Delta$. Now consider a player type h. If he demands one unit, the auction either ends immediately (if the rival also demands one unit), and otherwise ends one round later, at price $p = \Delta$, by 1). Whereas, if he demands two units, we can show that he also gets only unit, but pays either $p = \Delta$ or $p = 2\Delta$. Therefore, demanding one unit is a weakly dominant strategy. It remains to be shown that demanding two units has the asserted consequence. If he demands two units, the auction moves to the second round, either with state $(2, 1)$ or $(2, 2)$. If the state is $(2, 1)$, he will reduce demand to one unit, and thus end the auction at price $p = \Delta$. If the state is $(2, 2)$, both players know with certainty, that they are both type h, because for type l demanding one unit is a weakly dominant strategy, as we showed above. Therefore, in that event, it is a dominant strategy for both players to reduce demand to one unit, which ends the auction at price $p = 2\Delta$. Therefore, a player type h obtains one unit with certainty, and pays either $p = \Delta$ or $p = 2\Delta$, as asserted. This completes the proof. □

4 Equilibria when types are "very" different

We now turn to the case when the difference between $v_h(2)$ and $v_l(1)$ is "large". There, the following critical values of the prior belief play a crucial role:

$$\hat{\mu} := \frac{v_h(2) - (2p_l + 2\Delta)}{v_h(2) - p_l}, \quad \bar{\mu} := \frac{\Delta}{v_l}, \quad \check{\mu} := \frac{\Delta}{v_l - \Delta}. \tag{1}$$

PROPOSITION 2 *If $v_h(2) \geq 2p_l + \Delta$, the game has a pooling equilibrium for $\mu_0 \geq \hat{\mu}$, a separating equilibrium (that involves at most a one–round delay) for $\mu_0 \leq \check{\mu}$, and purely mixed strategy equilibria otherwise.*

PROOF 1) We first establish that we get a pooling equilibrium for $\mu \geq \hat{\mu}$. For that purpose, assume that the rival plays the pooling strategy, and demands one unit, regardless of his type. Consider a player type l. Obviously, his best reply is to also demand one unit. Next, consider a player type h. If he also demands one unit, the auction ends immediately, and his payoff is equal to $\pi^1 := v_h(1)$, whereas, if he unilaterally deviates, the best he can do is to demand two units up to price $p = p_l + \Delta$, and thereafter reduce demand if the rival has not dropped out of the auction (which occurs if the rival is h) and otherwise maintain a demand for two units. In that case, his payoff is equal to

$$\pi^2 := (1 - \mu_0)\left(v_h(1) + v_h(2) - 2(p_l + \Delta)\right) + \mu_0\left(v_h(1) - (p_l + 2\Delta)\right).$$

As one can easily confirm, $\pi^1 \geq \pi^2 \iff \mu_0 \geq \hat{\mu}$, as asserted.

2) We now show that the game does not have a "delayed separating equilibrium" that involves more than one round of delay. Suppose we have such an equilibrium, that involves a delay of $k > 1$ rounds. Then, a bidder type l is better off if he deviates and bids one unit in the first round, because that increases his payoff from $\pi_l^* = (1 - \mu_0)(v_l(1) - k\Delta)$ to $\pi_l = (1 - \mu_0)(v_l(1) - \Delta)$. Therefore, if the game has a delayed separating equilibrium, it involves at most a one–round delay.

3) We now compute the set of priors for which the game has either a separating equilibrium or a one–round delayed separating equilibrium.

For that purpose, assume first that the rival plays according to the candidate delayed separating equilibrium, and, in the first round demands two units, and switches to the separating equilibrium strategy after one round. Consider a player type l. If he also plays according to the delayed separating equilibrium, he obtains one unit if and only if the rival is also type l, which leads to the payoff: $\pi_l^1 = (1 - \mu_0)(v_l(1) - \Delta)$. Whereas, if he deviates, and demands two units in rounds 1 and 2, he thus convinces his rival that he is type h, and then both players demand one unit in the third round, which leads to the payoff: $\pi_l^2 = v_l(1) - 2\Delta$. Obviously, that deviation does not pay if and only if $\mu_0 \leq \check{\mu}$. Next, consider a player type h. If he sticks to the candidate equilibrium, his payoff is $\pi_h^2 = (1-\mu_0)\left(v_h(1) + v_h(2) - 2(p_l + \Delta)\right) + \mu_0(v_l(1) - 2\Delta)$. Whereas, if he deviates, and demands only one unit in round 1 or round 2, he obtains one unit for sure, but either pays the price $p = \Delta$ (in the event when the rival is type l) or $p = p_l + 2\Delta$ (otherwise), which leads to the payoff: $\pi_h^1 = (1-\mu_0)(v_h(1) - \Delta) + \mu_0\left(v_h(1) - (p_l + 2\Delta)\right)$. Evidently, that deviation does never pay, by the assumption $v_h(2) \geq 2p_l + \Delta$. Therefore, a delayed separating equilibrium exists for all $\mu_0 \leq \check{\mu}$.

Next, we compute the set of priors for which the game has an undelayed separating equilibrium. For that purpose, assume that the rival plays according to the candidate undelayed separating equilibrium (see (**??**)), and, in the first round, demands one unit if he is type l and two units otherwise. Consider a player type l. If he also plays according to the separating equilibrium, he obtains one unit if and only if the rival is also type l, which leads to the payoff: $\pi_l^1 = (1-\mu_0)v_l(1)$. Whereas, if he deviates, and demands two units in round 1, he thus convinces his rival that he is type h, and then both players demand one unit in the following round, which leads to the payoff: $\pi_l^2 = v_l(1) - \Delta$. Obviously, that deviation does not pay if and only if $\mu_0 \leq \bar{\mu}$. Next, consider a player type h. If he sticks to the candidate equilibrium, his payoff is $\pi_h^2 = (1-\mu_0)\left(v_h(1) + v_h(2) - 2(p_l + \Delta)\right) + \mu_0(v_l(1) - \Delta)$. Whereas, if he deviates, and demands only one unit in round 1, he obtains one unit for sure, but either pays the price $p = 0$ (in the event when the rival is type l) or $p = p_l + 2\Delta$ (otherwise), which leads to the payoff: $\pi_h^1 = v_h(1) - \mu_0(p_l + 2\Delta)$. Evidently, that deviation does not pay if and only if

$$\mu_0\left(v_h(2) - 3(p_l + \Delta)\right) \leq v_h(2) - 2(p_l + \Delta). \tag{2}$$

If $v_h(2) \geq 2(p_l + \Delta)$, then condition (2) is always satisfied. Therefore, in that case an undelayed separating equilibrium exists for all $\mu_0 \leq \bar{\mu}$. However, if $v_h(2) < 2(p_l + \Delta)$, then condition (2) takes the form

$$\mu_0 \geq \frac{2(p_l + \Delta) - v_h(2)}{3(p_l + \Delta) - v_h(2)} =: \tilde{\mu}. \tag{3}$$

Therefore, in that case, an undelayed separating equilibrium exists if and only if $\mu_0 \leq \bar{\mu}$ and $\mu_0 \geq \tilde{\mu}$, which happens to be an empty set if $\tilde{\mu} > \bar{\mu}$.

Obviously, $\check{\mu} > \bar{\mu}$. Therefore, we conclude that the game has a (delayed) separating equilibrium for all $\mu \leq \check{\mu}$ and an undelayed separating equilibrium for proper subset of that parameter set.

4) The game is finite. Therefore, it has an equilibrium, for all prior beliefs. Since we have already ruled out delayed separating equilibria that involve more than one round of delay, the equilibrium must be either pooling or

separating or mixed. Therefore, for all priors for which we have neither a pooling nor a (at most one–round delayed) separating equilibrium, the game has an equilibrium in purely mixed strategies. □

COROLLARY 1 *If Δ is made arbitrarily small, separating equilibria vanish, pooling equilibria exist for $\mu \geq 1 - \frac{v_l(1)}{v_h(2)-v_l(1)}$, and mixed strategy equilibria for all other μ.*

EXAMPLE 1 *Suppose $v_h(2) = 4, v_l(1) = 1, \Delta = \frac{1}{3}$. Then, the equilibrium is pooling if $\mu_0 \geq \frac{4}{9}$, undelayed separating if $\mu_0 \leq \frac{1}{3}$, and (one–round) delayed separating for all $\mu_0 \leq \frac{1}{2}$. (This is independent of the size of $v_h(1)$.)*

If Δ is reduced to $\Delta = \frac{1}{4}$, then the equilibrium is pooling if $\mu_0 \geq \frac{1}{2}$, undelayed separating if $\mu_0 \leq \frac{1}{4}$, and (one–round) delayed separating for all $\mu_0 \leq \frac{1}{3}$, and mixed otherwise. (This is independent of the size of $v_h(1)$.)

5 Conclusions

In the present paper we analyzed an English clock auction with two bidders, two types, and two units, assuming decreasing marginal valuations. That auction game is a sequential signaling game. We showed that it has a unique equilibrium, when bidders' types are "not very" different. In that equilibrium, both bidders play a pooling strategy, and the auction ends immediately at price $p = 0$. However, when the two types of bidders are "very" different, the low price equilibrium is preserved only if the prior belief μ_0 is sufficiently high, whereas otherwise the price goes up to the competitive price with positive probability. Interestingly, if the price increment goes to zero, the game has neither a delayed nor an undelayed separating equilibrium.

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