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EXPOSITA NOTE

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Immediate demand reduction in simultaneous ascending-bid auctions: a uniqueness result

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Abstract The present note analyzes the simultaneous ascending-bid auction with arbitrarily many asymmetric bidders with decreasing marginal valuations under complete information. We show that the game is solvable by iterated elimination of weakly dominated strategies if the efficient allocation assigns at least one unit to every player and if bid increments are sufficiently small. In that unique equilibrium, bidders immediately reduce their demand to the efficient allocation, and the auction ends in the first round of bidding.

Keywords Simultaneous ascending–bid auction · Weak dominance · Multi–unit auctions · Game theory

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1 Introduction

Simultaneous ascending-bid auctions have become an important device for allocating multiple units of goods.[1] They are also an important theoretical benchmark for models of price formation and resource allocation.[2] Therefore, the deserve a thorough game theoretic analysis.

The present note analyzes the simultaneous ascending-bid auction with arbitrarily many asymmetric bidders with decreasing marginal valuations under complete information. We show that the game is solvable by iterated elimination of weakly dominated strategies if the efficient allocation assigns at least one unit to every player and if bid increments are sufficiently small. In that unique equilibrium, bidders immediately reduce their demand to the efficient allocation, and the auction ends immediately. We also show by examples that the assumptions we make are necessary.

The assumptions that ensure uniqueness are fairly weak. The only economically significant assumption is that every bidder gets at least one object in the efficient allocation. The need for this assumption is intuitively plausible. Demand reduction pays only for those who will obtain a positive payoff. Players who anticipate that they will get nothing have the weakly dominant strategy to bid up to their marginal valuation for the first unit, and thus they drive up prices. In this case, it may even pay for the stronger bidders to yield one unit to the weak bidder in order to keep prices low (see Example 1 below). On the other hand, if prices go up and all the players who get nothing in the efficient allocation have dropped out, our analysis applies again and the auction stops immediately at the prevailing price.

The practical relevance of demand reduction has been confirmed in a number of simultaneous ascending-bid auctions. For example, Weber (1997) shows that demand reduction was at work in several FCC spectrum auctions. And Grimm et al. (2003) report the particularly spectacular case of the second generation GSM–spectrum auction in Germany, where the two dominant market players immediately reduced their demand to one half of the available radio spectrum, and thus immediately ended the auction, after they had succeeded to crowd out the two weaker bidders.

The theoretical incentive for strategic demand reduction is also well known.[3] However, it seems to be less well known that the low-price equilibrium is the unique equilibrium that survives iterated elimination of weakly dominated strategies even if there are many bidders.

The papers most closely related are Grimm et al. (2003), where we prove uniqueness for the case of two identical bidders, and the frequently cited paper by Ausubel and Schwartz (1999), which also reports a uniqueness result. However, by assuming that bidding follows a given order of moves, the paper by Ausubel and Schwartz (1999) analyzes a finite, sequential bargaining problem with declining

[1] Starting with radio spectrum auctions for the FCC, variants of the simultaneous ascending Auction have been applied in various fields as in electricity, gas, and environmental markets (see e.g. Cramton (2005) or Milgrom (2004) for recent accounts).

[2] Cf. Milgrom (2000).

[3] See Cramton (2005), Ausubel and Milgrom (2002), Ausubel and Cramton (2002), and Menezes (1996) for dynamic auctions, and Back and Zender (1993) and Wilson (1979) for sealed-bid auctions, and Engelbrecht-Wiggans and Kahn (1998) and Brusco and Lopomo (2002) for increasing or flat marginal valuations.

discount factors, rather than an auction.[4] Adopting a backward induction argument in the spirit of Ståhl (1972) and Rubinstein (1982), yields uniqueness of low price equilibrium. In a typical auction bids are submitted simultaneously. This makes it impossible to invoke results from bargaining theory to establish uniqueness, because in simultaneous bargaining even strong refinements do not imply uniqueness (see Chatterjee and Samuelson 1990). Another difference is that Ausubel and Schwartz (1999) assume two bidders with flat valuations, whereas we consider an arbitrarily large number of bidders with declining valuations.

2 Model

There are $M \geq 2$ bidders who bid for $N \geq 2$ objects. Bidders' marginal valuations, $v_k^i, i = 1, \ldots, M, k = 1, \ldots, N$, are strictly decreasing in the number of units k. $w_k^i := \sum_{l=1}^{k} v_l^i$ is the absolute valuation of bidder i for k objects.

The auction is an open, ascending-bid clock auction. The price clock goes up by the fixed increment $\Delta > 0$, starting at 0, until there is no excess demand. In each round $t = 0, 1, 2, \ldots$, bidders simultaneously submit a bid $\beta^i(t) \in \{0, 1, 2, \ldots, N\}$, $i = 1, \ldots, M$, which states how many units they demand at the given price $t\Delta$. If the sum of bids does not exceed the supply N, the game ends in round t and bidders pay $t\Delta$ for each of the $\beta^i(t)$ objects they get.[5]

For later use, we introduce the demand function of bidder i in round t given by $d^i(t) = \max\{k : v_k^i \geq t\Delta\}$ if $v_N^i \geq t\Delta$ and $d^i(t) = 0$ else. Aggregate demand is $D(t) = \sum_{i=1}^{M} d^i(t)$.

Assumption 1 *All marginal valuations are distinct: for all* $(i, k) \neq (j, l)$*, we have* $v_k^i \neq v_l^j$*. The price increment is sufficiently small:* $0 < \Delta < \Delta^* := \min\left\{\frac{1}{N}\left(v_k^i - v_l^j\right) : v_k^i > v_l^j, i, j = 1, \ldots, M, k, l = 1, \ldots, N\right\}$*. Marginal valuations do not lie on the price grid: for all* $t = 0, 1, 2, \ldots$*,* $i = 1, \ldots, M$ *and* $k = 1, \ldots, N$ *we have* $v_k^i \neq t\Delta$*.*

Note that the assumption of pairwise distinct marginal valuations holds true generically. It is made here to avoid irrelevant case distinctions. Also, the particular value of Δ^* plays no role. What matters only is that the increment is sufficiently small.

Assumption 1 ensures that the price clock ticks are in between all marginal valuations. Thus, demand cannot have jumps bigger than one. As a consequence, there is a competitive solution where demand equals supply because aggregate demands starts at $D(0) = MN$, goes to zero as $t \to \infty$, and has no jumps greater than 1. The smallest competitive price on the price grid determined by Δ is given by $T^*\Delta$ with $T^* := \min\{t : D(t) = N\}$. The unique efficient allocation is $a^i := d^i(T^*)$, $i = 1, \ldots, M$.

[4] In the last rounds of an open, ascending-bid auction, bids are often made in repeating sequence. However, a bidder can always break that sequence by overbidding his own bid. Therefore, bidding is never trapped in a fixed sequence of moves.

[5] Bidding strategies may depend on the history of the game, of course. In order to keep the notation simple, we do not make this dependence explicit.

3 Uniqueness of equilibrium

Theorem 1 *Assume that the efficient allocation assigns at least one object to each player: $a^i > 0$ for all $i = 1, \ldots, M$. Under Assumption 1, the game can be solved by iterated elimination of weakly dominated strategies. In that solution, the efficient allocation (a^i) is implemented and the game ends in round 0 at price 0. The associated equilibrium strategy is $\beta^i(t) = \min\{a^i, d^i(t)\}$.*

Proof Denote by $T^0 = \min\{t : D(t) = 0\}$ the first round when aggregate demand is zero. For every $s \geq T^0$, the weakly dominant strategy in the subgame that starts at s is $\beta^i(t) = 0$ for all $t \geq s$. We thus eliminate all strategies except bidding zero in rounds $t \geq T^0$.

We now proceed in two steps. In Step 1, we show the only strategy that survives iterated elimination of weakly dominated strategies in subgames between T^* and T^0 prescribes to bid one's demand. In Step 2 below, we show that in the early rounds $t = 0, \ldots, T^*$, bidders reduce demand to their efficient quantity a^i.

Step 1 ($T^* \leq t \leq T^0$). We proceed by backward induction. For $t = T^0$, we know already that the only strategies that survive have $\beta^i(s) = d^i(s) = 0$ for $s \geq T^0$. Assume now that for all $s \geq t + 1$, we have eliminated all strategies that do not have $\beta^i(s) = d^i(s)$. In round $t \geq T^*$, bidding more than one's demand is weakly dominated by bidding one's demand. Hence, we can eliminate all strategies that have $\beta^i(t) > d^i(t)$. After this elimination, we know that the game ends in round t because $\sum_{i=1}^{M} \beta^i(t) \leq D(t) \leq D(T^*) = N$. In that case, bidders are just price takers, and take their optimal quantity, $\beta^i(t) = d^i(t)$.

Step 2 ($0 \leq t \leq T^*$). We show by backward induction that only $\beta^i(t) = a^i$ survives for $t = 0, \ldots, T^*$. We know by Step 1 that the claim is true for $t = T^*$ since $\beta^i(T^*) = d^i(T^*) = a^i$. Assume now that for all $s \geq t + 1$, $s \leq T^*$, we have eliminated all strategies that do not have $\beta^i(s) = a^i$. Our first claim is that it is weakly dominated to bid $b < a^i$. A bid $b < a^i$ can only lead to a higher payoff if the game ends in round t with the bid b and ends in round $t + 1$ with the bid a^i. In that case, the bid b yields a payoff $\pi(b) := w^i(b) - bt\Delta$, whereas the bid a^i leads to a payoff $\pi^* := w^i(a^i) - a^i(t + 1)\Delta$. Since the demand of player i is at least a^i in round t, we have $\pi(b) \leq \pi(a^i - 1) = w^i(a^i - 1) - (a^i - 1)t\Delta$. Hence, it is enough to show that $g := \pi^* - \pi(a^i - 1) > 0$. Note that by definition of T^* there exists a player j with $(T^* - 1)\Delta < v^j_{a^j+1} < T^*\Delta$. Since $t < T^*$, we also have $t\Delta < v^j_{a^j+1}$. By definition of a^i and Assumption 1, $T^*\Delta < v^i_{a^i}$. We now get $g = v^i_{a^i} - t\Delta - a^i\Delta > v^i_{a^i} - v^j_{a^j+1} - a^i\Delta$. By Assumption 1, we have $v^i_{a^i} - v^j_{a^j+1} - a^i\Delta \geq v^i_{a^i} - v^j_{a^j+1} - a^i\Delta^* > 0$. Hence $g > 0$ follows. This shows that bidding less than a^i is dominated by bidding a^i.

We thus eliminate all strategies with $\beta^i(t) < a^i$. Given this, every bidder plays at least his efficient quantity a^j, $j = 1, \ldots, M$. After this reduction, it is weakly dominant to play a^i in round t because bidding more just brings the game to the next round where it ends with the allocation (a^j). □

Example 1 The theorem does not hold true if some players do not get objects in the efficient allocation. Consider the following example[6] with two players and two

[6] Similar examples can be found in Ausubel and Milgrom (2002) and Milgrom (2000).

Table 1 Reduced payoff matrix in round 30

Game of example 2

	1	2	3
1	41, 43	41, 46	41, −1
2	72, 43	72, 46	68, 42
3	73, 43	68, 42	68, 42

objects. Valuations are $v_1^1 = 101$, $v_2^1 = 51$, $v_1^2 = 47$, $v_2^2 = 1$, and the increment is $\Delta = 2$. The efficient allocation is $a^1 = 2$, $a^2 = 0$. In this case, the strategy "truthful bidding" $\beta^2(t) = d^2(t)$ cannot be eliminated because player 2 has no incentive to reduce demand to his competitive allocation in early rounds here because he gets nothing in the efficient allocation. Dropping out does not weakly dominate bidding one's demand. On the contrary, for prices $p = 2, 4, 6, \ldots, 46$ player 2's weakly dominant strategy is to bid 1. Given this strategy, it is optimal for player 1 to reduce demand to 1 in round 0 because he thus obtains a payoff of 101, whereas he can get only 56 by driving prices up to 48. Knowing that outcome, it is weakly dominant to bid 1 in round 0 for both bidders. Note that the resulting allocation is not efficient but the equilibrium price is still 0.

On the other hand, if we increase the marginal valuation for the second object of the strong bidder sufficiently, say $v_2^1 = 97$, a similar argument shows that the unique outcome that survives iterated elimination of weakly dominated strategies implements the efficient allocation at the competitive price 48. The resulting allocation is efficient but the price is not 0. However, if one introduces participation fees or bidding costs in this kind of example, the weak bidder drops out immediately, and we get again zero prices.

Example 2 The assumption $\Delta \leq \Delta^*$ is necessary. Consider a two-player auction with four objects and valuations $v_1^1 = 101$, $v_1^2 = 91$, $v_3^1 = 61$, $v_4^1 = 11$, and $v_1^2 = 103$, $v_2^2 = 63$, $v_3^2 = 13$, and $v_4^2 = 1$. Here, $\Delta^* = \frac{1}{2}$. Now consider the increment $\Delta = 2$. The efficient allocation assigns two objects to each player. The competitive price is 62. At and above this price, every player plays his demand. However, at price $p = 60$ in round $t = 30$, bidding 1 is not weakly dominated by bidding 2 for player 2. The payoff matrix is shown in Table 1 (neglecting the bid 0). The game has two strict Nash equilbria: (3, 1) and (2, 2).

4 Conclusions

The present paper has not only shown that low-price outcomes may be an equilibrium in multi-unit auctions. We showed that the low–price equilibria that implement the efficient allocation at the minimum bid is rationalizable as the only equilibrium that survives iterated elimination of dominated strategies. This uniqueness holds true generically, provided two fairly weak requirements are satisfied: the increment by which the price clock goes up must be sufficiently small, and all active bidders' valuations must be such that each gets at least one unit in the competitive allocation.

References

Ausubel, L., Schwartz, J.: The ascending auction paradox. Working Paper, University of Maryland (1999)
Ausubel, L., Milgrom, P.: Ascending auctions with package bidding. In: Frontiers of theoretical economics, Vol. 1, pp. 1–42. Berkely, CA: Berkely Economic Press 2002
Ausubel, L., Cramton, P.: Demand reduction and inefficiency in multi-unit auctions. Working Paper, University of Maryland (2002)
Back, K., Zender, J.: Auctions of divisible goods. Rev Financ Stud **6**, 733–764 (1993)
Brusco, S., Lopomo, G.: Collusion via signalling in open ascending auctions with multiple objects and complementarities. Rev Econ Stud **69**, 1–30 (2002)
Chatterjee, J., Samuelson, L.: Perfect equilibria in simultaneous offers bargaining. Int J Game Theory **19**, 237–267 (1990)
Cramton, P.: Simultaneous ascending auctions combinatorial auctions. In: Cramton, P., Shoham, Y., Steinberg, R. (eds) Combinatorial auctions, Ch. 4. Cambridge: MIT Press 2005
Engelbrecht-Wiggans, R., Kahn, C.: Low revenue equilibria insimultaneous auctions. Working Paper, University of Illinois (1998)
Grimm, V., Riedel, F., Wolfstetter, E.: Low price equilibrium in multi-unit auctions: the GSM spectrum auction in Germany. Int J Ind Organ **21**(10) 1557–1569 (2003)
Menezes, F.M.: Multiple-unit English auctions. Eur J Polit Econ **12**, 671–684 (1996)
Milgrom, P.: Putting auction theory to work: the simultaneous ascending auction. J Polit Econ **108**, 245–272 (2000)
Milgrom, P.: Putting auction theory to work. Cambridge: Cambridge University Press 2004
Rubinstein, A.: Perfect equilibrium in a bargaining model. Econometrica **50**(1) 97–109 (1982)
Ståhl, I.: Bargaining theory. Stockholm, EFI The Economic Research Institute (1972)
Weber R.J.: Making more from less: strategic demand reduction in the FCC spectrum auctions. J Econ Manage Strategy **6** 529–548 (1997)
Wilson, R.: Auctions of shares. Q J Econ **93**, 675–689 (1979)