Prizes and Lemons: Technical Supplement

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In this technical supplement to Ding and Wolfstetter (2011) we provide some more detailed proofs and add material for the interested reader. In particular, we provide a more detailed characterization of payoff functions in fixed-prize tournaments, a characterization of optimal fixed-prize tournaments, and more results concerning the ranking of the optimal auction relative to the optimal simple fixed-prize tournament.

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1 Introduction

In this technical supplement to the paper we provide some more detailed proofs and add material for the interested reader. In particular, we provide a more detailed characterization of payoff functions in fixed-prize tournaments, a characterization of optimal fixed-prize tournaments, and more results concerning the ranking of the optimal auction relative to the optimal simple fixed-prize tournament.

2 Supplement to the proof of Lemma 6

Suppose both innovators have registered or no registration is required. In Lemmas 5and Lemma 6 in the paper we showed already that the game played between innovators has an equilibrium in cutoff strategies and that innovators submit all innovations if and only if $p \geq \bar{p}$. However, there we did not prove that the cutoff strategy γ is strictly monotone increasing in p for all $p < \bar{p}$. Here we we fill in this gap.

To prepare the proof and solve the symmetric equilibrium cutoff strategy γ, consider one player, say player 1, who contemplates the deviating strategy $\gamma_1 \geq \gamma$, while his rival, player 2, plays the equilibrium strategy γ. To compute the payoff function of player 1, $\pi_1(\gamma_1, \gamma)$, take a look at the state space representation of that innovator's payoffs in Figure 1. Using the joint density $g_{12}(x_1, x_2) = g(x_1)g(x_2)$, one can then

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compute the payoff function by integrating over the relevant subsets of the state space $[0,1]\times[0,1]$:

$$\begin{aligned}\pi_1(\gamma_1,\gamma) = p\Bigg(&\int_0^{\gamma}\int_0^{x_1} dG(x_2)dG(x_1) + \int_{\gamma}^{\gamma_1}\int_0^{\gamma} dG(x_2)dG(x_1)\\ &+\int_0^{\gamma_1}\int_{\gamma}^{1} dG(x_2)dG(x_1)\Bigg) + \frac{1}{2}\int_{\gamma_1}^{1}\int_0^{\gamma}(x_1-x_2)\,dG(x_2)dG(x_1)\\ &+\frac{1}{2}\int_{\gamma_1}^{1}\int_{\gamma}^{x_1}\left(x_1-\frac{1}{2}x_2\right)dG(x_2)dG(x_1)-c\end{aligned}\tag{1}$$

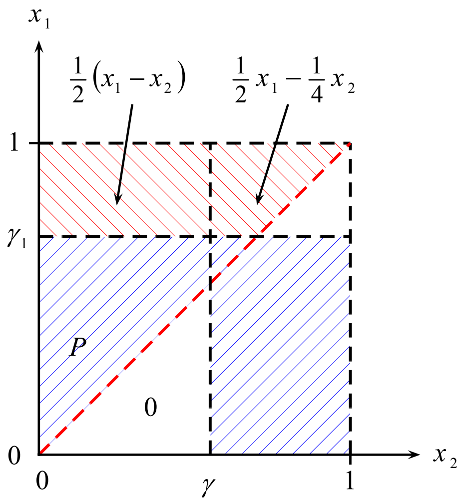

Figure 1: Payoffs of innovator 1 for $\gamma_1 \geq \gamma$ in the state space $[0,1]\times[0,1]$

Proposition 1. *In a tournament with fixed price p, suppose both innovators have registered or no registration is required and assume $p < \bar{p}$. The game played between innovators has a unique symmetric equilibrium strategy $\gamma \in (0,1]$, which is implicitly defined as the solution of*

$$p = \frac{1}{2}\int_0^{\gamma} G(x)dx. \tag{2}$$

γ is strictly increasing in p for all $0 < p < \bar{p}$.

Proof. Consider one innovator, say innovator 1. We need to show that for each given p, the γ implicitly defined in Proposition 1 satisfies the equilibrium requirement

$$\gamma = \arg\max_{0\leq\gamma_1\leq 1}\pi_1(\gamma_1,\gamma). \tag{3}$$

For this purpose, first consider "upward" deviations from the equilibrium, $\gamma_1 \geq \gamma$, as in (1). Computing the partial derivative of π_1 w.r.t. γ_1 gives

$$
\begin{aligned}
\frac{\partial \pi_1}{\partial \gamma_1} &= p\left(G(\gamma)g(\gamma_1)+(1-G(\gamma))g(\gamma_1)\right) \\
&\quad -\frac{1}{2}g(\gamma_1)\int_0^{\gamma}(\gamma_1-x_2)dG(x_2) \\
&\quad -g(\gamma_1)\int_{\gamma}^{\gamma_1}\left(\frac{1}{2}\gamma_1-\frac{1}{4}x_2\right)dG(x_2)
\end{aligned}
\tag{4}
$$

$$
\left.\frac{\partial \pi_1}{\partial \gamma_1}\right|_{\gamma_1=\gamma} = \left(p-\frac{1}{2}\int_0^{\gamma}G(x)dx\right)g(\gamma) =: \xi(p,\gamma)g(\gamma). \tag{5}
$$

Using the Lagrange function $\mathcal{L} := \pi_1 + \lambda(1-\gamma_1)$, with the Lagrangian λ, and invoking the equilibrium requirement that γ must be such that the best response of innovator 1 to γ is $\gamma_1 = \gamma$ (see (3)), the equilibrium strategy γ must solve the Kuhn-Tucker (KT) conditions

$$
\begin{aligned}
&\left.\frac{\partial \mathcal{L}}{\partial \gamma_1}\right|_{\gamma_1=\gamma} = \left.\frac{\partial \pi_1}{\partial \gamma_1}\right|_{\gamma_1=\gamma} - \lambda = 0 \\
&\left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_{\gamma_1=\gamma} = 1-\gamma \geq 0 \quad \text{and} \quad \left.\frac{\partial \mathcal{L}}{\partial \lambda}\right|_{\gamma_1=\gamma} \lambda = 0.
\end{aligned}
\tag{6}
$$

For $p \geq \bar{p}$ one finds (omitting the subscript 1) $\partial\pi/\partial\gamma|_{\gamma=1} \geq 0$; hence, the KT conditions are solved by $(\gamma = 1, \lambda = \partial\pi/\partial\gamma|_{\gamma=1})$. This confirms Lemma 6.

For $0 < p < \bar{p}$ one finds (omitting the subscript 1) $\partial\pi/\partial\gamma|_{\gamma=1} < 0$; hence, the KT conditions are solved by $(0 < \gamma < 1, \lambda = 0)$, where γ is implicitly defined as the unique solution of equation (2).

A similar argument deals with "downward" deviations, $\gamma_1 \leq \gamma$; it yields the same results.

Uniqueness of the solution for $p < \bar{p}$ follows from the fact that $\xi(p,\gamma)$ is strictly decreasing in γ and that $\gamma = 0 \Rightarrow \xi(p,\gamma) = p > 0$ and $\gamma = 1 \Rightarrow \xi(p,\gamma) = p - {}^1\!/_2(1 - E[X]) < 0$. Monotonicity of $\gamma(p)$ follows easily.

Finally, we show that the unique solution of the condition

$$
\left.\frac{\partial \pi_1}{\partial \gamma_1}\right|_{\gamma_1=\gamma} = \xi(p,\gamma)g(\gamma) = 0
$$

is indeed a maximizer of the payoff of innovator 1 (assuming innovator 2 also plays the strategy γ). We prove this by showing that the function $\pi_1(\gamma_1,\gamma)$ is pseudoconcave in γ_1.[1] For this purpose, compute the cross derivative, using (4):

$$
\frac{\partial^2}{\partial \gamma_1 \partial \gamma}\pi_1 = \frac{1}{4}\gamma g(\gamma_1)g(\gamma) \geq 0.
$$

Together with the monotonicity of $\xi(p,\gamma)$ in γ, it follows that

$$
\gamma_1 < \gamma \Rightarrow \frac{\partial}{\partial \gamma_1}\pi_1(\gamma_1,\gamma) \geq \frac{\partial}{\partial \gamma_1}\pi_1(\gamma_1,\gamma_1) = \xi(p,\gamma_1)g(\gamma_1) > \xi(p,\gamma)g(\gamma_1) = 0
$$

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[1] On pseudoconcavity see Avriel, Diewert, Schaible, and Zang (1988, p. 93 ff.).

$$\gamma_1 > \gamma \Rightarrow \frac{\partial}{\partial \gamma_1}\pi_1(\gamma_1,\gamma) \leq \frac{\partial}{\partial \gamma_1}\pi_1(\gamma_1,\gamma_1) = \xi(p,\gamma_1)g(\gamma_1) < \xi(p,\gamma)g(\gamma_1) = 0.$$

Therefore, $\pi_1(\gamma_1,\gamma)$ is increasing to the left of its stationary point (for $\gamma_1 < \gamma$) and decreasing to the right of its stationary point (for $\gamma_1 > \gamma$). Hence, the stationary point is a global maximum. ☐

3 Optimal fixed-prize tournament

In this section we show how one can compute the optimal simple fixed-prize tournament and illustrate it with an example.

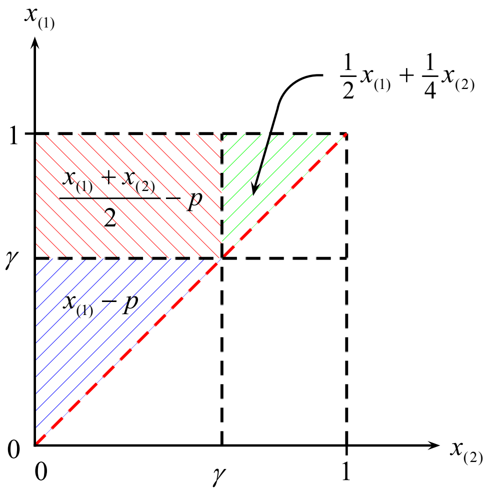

Figure 2: Payoffs of the procurer in the order statistics space

For this purpose, we compute the procurer's payoff as a function of γ, eliminating the variable p. For this task, take a look at Figure 2, where the procurer's profits are represented in the order statistics space. The joint p.d.f. of $X_{(1)}, X_{(2)}$ is $g_{(1,2)}(x,y) = 2g(x)g(y)$. Therefore, one obtains, after a bit of rearranging, for all $\gamma \leq 1$ (resp. $p \leq \bar{p}$):

$$\begin{aligned}\pi_p(\gamma) = {} & 2\int_0^\gamma \int_0^x (x-p)g(x)g(y)dydx + 2\int_\gamma^1 \int_0^\gamma \left(\frac{x+y}{2} - p\right) g(y)g(x)dydx \\ & + 2\int_\gamma^1 \int_\gamma^x \left(\frac{1}{2}x + \frac{1}{4}y\right) g(y)g(x)dydx.\end{aligned} \tag{7}$$

Whereas for $p \geq \bar{p}$ (resp. $\gamma = 1$) one has

$$\pi_p = E[X_{(1)}] - p. \tag{8}$$

The optimal fixed-prize tournament maximizes the procurer's expected profit over γ, resp. p, subject to the constraint that innovators' equilibrium expected payoff is nonnegative.

Example 1. *Suppose* $G(x) \equiv x$ *(uniform distribution) and* $c = \frac{1}{15}$. *Then,* $\bar{p} = \frac{1}{2}\int_0^1 y\,dy = \frac{1}{4}$, *innovators' equilibrium strategy is*

$$\gamma(p) = \begin{cases} 2\sqrt{p} & \textit{if} \quad p < 1/4 \\ 1 & \textit{if} \quad p \geq 1/4 \end{cases}$$

the procurer's payoff function, as a function of p, *is* $\pi_p(\gamma) = 5/12 + \gamma^2/4 - \gamma^3/2 + \gamma^4/4$, *the optimal fixed-prize tournament is*

$$\gamma^* = \arg\max_{\gamma} \pi_p(\gamma) = 1/2, \quad \textit{resp.} \quad p^* = 1/16,$$

and innovators' equilibrium payoff is $\pi^* = 15/128 - c \geq 0$ *(see Figure 3).*

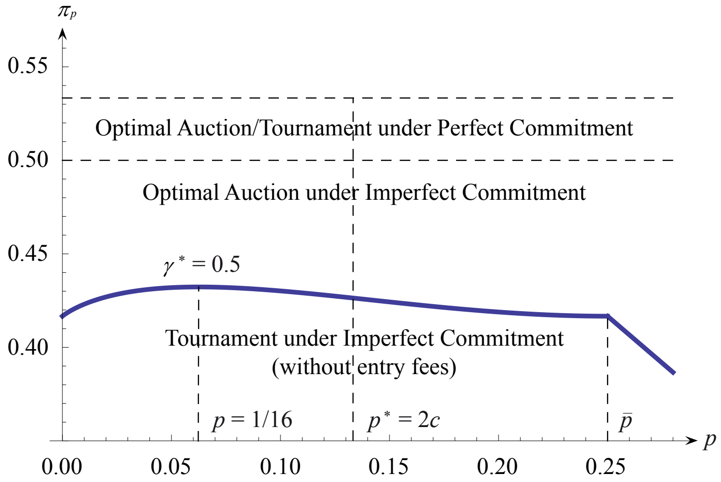

Figure 3: Comparing the optimal auction with the optimal fixed-prize tournament (without entry fees), assuming a uniform distribution

This example for a uniform distribution (see Example 1) is illustrated in Figure 3. There, the solid curve plots the procurer's expected profit in the fixed-prize tournament, as a function of the prize p. It has a kink at $p = \bar{p} = 1/4$ (which is the smallest prize that prevents bypass). The optimal prize is equal to $p = 1/16$, which is substantially lower than the optimal prize under perfect commitment ($p^* = 2/15$), and the optimal γ is equal to $\gamma^* = 1/2$. Therefore, in the optimal fixed-prize tournament, all innovations $X > 1/2$ bypass the tournament. Evidently, the lack of perfect commitment hurts the procurer in the fixed-prize tournament as well as in the auction.

Not surprisingly, the optimal auction performs better in this example, since a uniform distribution does not satisfy the requirement that $\eta > 1$ for superiority of tournaments stated in Proposition 4 from the paper.

4 Why the auction cannot be improved by requiring a minimum score

It has been shown in the theory of auctions that an optimal auction usually involves *either* an entry fee *or* a reserve price, and there is no benefit of employing both. However, unlike in the standard optimal auction problem, in our analysis the entry fee is levied before potential bidders draw their value. Thus, one may think that the profitability of an auction can be improved by adding a reserve bid requirement, following an entry fee.

In a first step we assure that the minimum score does not affect the existence of an equilibrium in cutoff strategies.

To ensure that the auction has a symmetric equilibrium in cutoff strategies, we assume that $K := -G$ is star-shaped, which is weaker than concavity but stronger than subadditivity of G. The function K is star-shaped if for each $\alpha \in [0,1]$, and all x: $K(\alpha x) \leq \alpha K(x)$ (see Bruckner and Ostrow, 1962). star-shapedness implies that $K(x)/x$ is increasing, resp. $G(x)/x$ is decreasing. This property is used in the proof below.

Suppose the procurer accepts only bids that match or exceed a stated minimum score, which is denoted by R. This changes the auction as follows: if exactly one bidder, say bidder 1, submits a score $S_1 = x_1 - b_1 \geq R$, that bidder wins the auction and is paid a price equal to $x_1 - R$ (instead of a price equal to $x_1 - x_2$); if no bidder submits a score equal to R or more, no trade occurs in the auction; and if both bidders submit a score $S \geq R$, the minimum score does not bind, and the auction proceeds as before. Of course, if a bidder does not submit a valid bid, he will try to engage in bargaining, after the auction.

In the presence of a minimum score requirement, bidders play cutoff strategies and bid if and only if the value of their innovation is equal or greater than a threshold value, which is denoted by r. We look for a symmetric equilibrium. In such an equilibrium, a bidder with value $x = r$ must be indifferent between submitting a score $S = R$ and not bidding, and bidding must be profitable for all $x > r$, and unprofitable for all $x < r$.

Indifference between bidding and not bidding for $x = r$ means that

$$G(r)(r-R) = \frac{1}{2}\int_0^r (r - \frac{1}{2}y)g(y)dy.$$

This implies the following unique and strictly increasing relationship between the minimum score R and the threshold value r

$$R = \frac{1}{2}r + \frac{1}{4}E[X \mid X \leq r], \tag{9}$$

which in turn allows us to eliminate the variable R, compute the procurer's expected profit as a function of r, and then maximize that payoff over r.

Next we prove that the procurer cannot increase his expected profit by adding a minimum score requirement.

Proof. Denote the difference between bidders' payoff when bidding and not bidding by Δ. Assume $x > r$. Then, using the relationship between R and r

$$\begin{aligned}\Delta &= G(r)(x-R) + \int_r^x (x-y)g(y)dy - \frac{1}{2}\int_0^r (x - \frac{1}{2}y)g(y)dy - \frac{1}{2}\int_r^x (x-y)g(y)dy \\ &= G(r)(x-R) + \frac{1}{2}\int_r^x (x-y)g(y)dy - \frac{1}{2}\int_0^r (x-\frac{1}{2}y)g(y)dy \\ &= G(r)x - G(r)r + \frac{1}{2}\int_0^r (r-\frac{1}{2}y)g(y)dy + \frac{1}{2}\int_r^x (x-y)g(y)dy - \frac{1}{2}\int_0^r (x-\frac{1}{2}y)g(y)dy \\ &= G(r)(x-r) + \frac{1}{2}\int_0^r (r-x)g(y)dy + \frac{1}{2}\int_r^x (x-y)g(y)dy\end{aligned}$$

$$
\begin{aligned}
&= G(r)(x-r) + \frac{1}{2}(r-x)G(r) + \frac{1}{2}\int_r^x (x-y)g(y)dy \\
&= \frac{1}{2}G(r)(x-r) + \frac{1}{2}\int_r^x (x-y)g(y)dy \\
&> 0.
\end{aligned}
$$

Similarly, one obtains for $x \leq r$:[2]

$$
\begin{aligned}
\Delta &= G(r)(x-R) - \frac{1}{2}\int_0^x (x-\frac{1}{2}y)g(y)dy \\
&= G(r)x - G(r)r + \frac{1}{2}\int_0^r (r-\frac{1}{2}y)g(y)dy - \frac{1}{2}\int_0^x (x-\frac{1}{2}y)g(y)dy \\
&= G(r)(x-r) + \frac{1}{2}rG(r) - \frac{1}{2}xG(x) - \frac{1}{4}\int_x^r yg(y)dy \\
&\leq G(r)(x-r) + \frac{1}{2}rG(r) - \frac{1}{2}xG(x) - \frac{1}{4}\int_x^r xg(y)dy \quad \text{(since } y \leq x\text{)} \\
&= G(r)(x-r) + \frac{1}{2}rG(r) - \frac{1}{2}xG(x) - \frac{1}{4}x(G(r)-G(x)) \\
&= \frac{3}{4}xG(r) - \frac{1}{4}xG(x) - \frac{1}{2}rG(r) \\
&\leq \frac{1}{2}xG(r) - \frac{1}{4}xG(x) - \frac{1}{2}rG(r) + \frac{1}{4}rG(x) \quad \text{(since } -G \text{ is star-shaped)} \\
&= \left(G(r) - \frac{1}{2}G(x)\right)\left(\frac{1}{2}x - \frac{1}{2}r\right) < 0
\end{aligned}
$$

The addition of a minimum score implies restrictions on the entry fee. For a given r the procurer sets the highest entry fee that ensures that both innovators register. Consider an innovator whose rival registers. Denote his payoff if he also registers by π^r and if he does not register by π^n. Then, the procurer sets the highest fee that ensures $\pi^r \geq \pi^n$ and $\pi^r \geq 0$.

After some rearranging and changing the order of integration one finds

$$
\begin{aligned}
\pi^r &= \frac{1}{2}\int_0^r \int_0^x (x-\frac{1}{2}y)g(x)g(y)dydx + \int_r^1 \int_r^x (x-y)g(x)g(y)dydx \\
&\quad + \int_r^1 \int_0^r (x-R)g(x)g(y)dydx - f - c \\
\pi^n &= \frac{1}{2}\int_0^r \int_y^1 (x-\frac{1}{2}y)g(x)g(y)dxdy + \frac{1}{2}\int_r^1 \int_y^1 (x-y)g(x)g(y)dxdy - c.
\end{aligned}
$$

The highest entry fee, f^{**}, that ensures $\pi^r \geq \pi^n$ is

$$
f^{**} := \int_r^1 \int_r^x (\frac{1}{2}x - \frac{1}{2}y)g(x)g(y)dydx + \int_r^1 \int_0^r (\frac{1}{2}x - \frac{1}{4}y - R)g(x)g(y)dydx.
$$

And the highest entry fee, f^*, that ensures $\pi^r \geq 0$ is

$$
f^* := \frac{1}{2}\int_0^r \int_0^x (x-\frac{1}{2}y)g(x)g(y)dydx
$$

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[2] Note, for some interval of x values below r one has nevertheless $x > R$.

$$+\int_r^1\int_r^x (x-y)g(x)g(y)dydx+\int_r^1\int_0^r (x-R)g(x)g(y)dydx-c.$$

Therefore, the optimal entry fee is $f=\min\{f^{**},f^*\}$.

Finally, compute the procurer's expected profit, using the optimal registration fee and the relationship between R and r, writing π_p as a function of r. If $f^{**}\le f^*$, one finds

$$\begin{aligned}\pi_p &= 2\int_0^r\int_0^x (\frac{1}{2}x+\frac{1}{4}y)g(x)g(y)dydx+2\int_r^1\int_r^x yg(x)g(y)dydx\\ &+2\int_r^1\int_0^r Rg(x)g(y)dydx+2f^{**}\\ &=\int_0^1\int_0^x (x+\frac{1}{2}y)g(x)g(y)dydx+\int_r^1\int_r^x \frac{1}{2}yg(x)g(y)dydx,\end{aligned}$$

which is decreasing in r and thus reaches the maximum at $r=0$, associated with $R=0$. Thus, in this case, the procurer cannot benefit from including a minimum score requirement.

Similarly, if $f^*\le f^{**}$,

$$\begin{aligned}\pi_p &= 2\int_0^r\int_0^x (\frac{1}{2}x+\frac{1}{4}y)g(x)g(y)dydx+2\int_r^1\int_r^x yg(x)g(y)dydx\\ &+2\int_r^1\int_0^r Rg(x)g(y)dydx+2f^{*}\\ &=\int_0^1\int_0^x 2xg(x)g(y)dydx-2c\\ &=E[X_{(1)}]-2c.\end{aligned}$$

Since $E[X_{(1)}]-2c$ is the procurer's expected profit in the auction without minimum score, it follows also in this case that the procurer cannot benefit from a minimum score requirement. □

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