A Note on Bayesian Nash Equilibria in Imperfectly Discriminating Contests

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Abstract

The literature on imperfectly discriminating contests has almost exclusively focused on complete information. We study such contests assuming players have private information. We identify a general class of imperfectly discriminating contests for which findings by Athey (2001) imply the existence of a Bayesian Nash equilibrium in monotone pure strategies. The main assumptions are that a player's valuation is increasing in the signal he observes and that a player's probability of winning is continuous in the efforts of all players as well as increasing in his own effort.

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1 Introduction

Models of contests have been used to study a wide range of relevant economic questions.[1] In many contest-like situations players are, at least to some extent, privately informed about their preferences. Depending on whether the contestant who invested the highest effort wins with certainty or not, one can distinguish between perfectly and imperfectly discriminating contests. Many important applications employ contests of the latter kind, as for example models of rent-seeking (Tullock, 1980) or labor market tournaments (Lazear and Rosen, 1981).

Perfectly discriminating contests such as the all-pay auction and the war of attrition have been thoroughly studied under incomplete information (e.g., Krishna and Morgan, 1997). The literature on imperfectly discriminating contests, however, has almost exclusively focused on completely informed contestants. Notable exceptions include Hurley and Shogren (1998), Malueg and Yates (2004), and Fey (2008) who study two-player contests in the independent private values framework, making very specific assumptions concerning the distribution types are drawn from and the contest success function (CSF).[2]

The information structure and the CSF we consider in this note are much less restrictive. We identify a general class of imperfectly discriminating contests with private information to which a well-known result by Athey (2001) from the theoretical literature on Bayesian games can be applied, establishing existence of a Bayesian Nash equilibrium in monotone pure strategies. An important requirement for this approach is that the CSF is everywhere continuous. Apart from that we make only weak assumptions, allowing for interdependent valuations. We require a player's valuation to be increasing in the signal he observes and his probability of winning to be increasing in his effort. Most notably, no assumptions regarding concavity and convexity, respectively, of the contest success, valuation, and cost function are needed for the existence result.

We proceed by stating the assumptions and proving the main result. This is fol-

[1] See, e.g., Konrad (2009) for a recent survey.

[2] They consider Tullock lottery contests where a player's probability of winning is equal to his effort divided by the sum of all efforts. Types are either drawn from simple discrete distributions or a continuous uniform distribution. Ryvkin (2010) and Wasser (2012) extend the analysis to more than two players and more general continuous distributions. Another branch of the literature considers one-sided asymmetric information (e.g., Wärneryd, 2003).

lowed by a short discussion of some examples from the contest literature our result applies to.

2 Model and Main Result

There are n risk neutral players competing for a single prize in a contest. Before the contest, each player i privately observes a signal $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}$. It is common knowledge among players that each θ_i is independently drawn from a continuous distribution $F_i(\theta_i)$ with bounded and atomless density $f_i(\theta_i)$. Let $\theta := (\theta_1, \dots, \theta_n)$ denote the vector of all signals. The value of the prize to player i can be expressed as a function $v_i : [\underline{\theta}_1, \overline{\theta}_1] \times \cdots \times [\underline{\theta}_n, \overline{\theta}_n] \to [\underline{v}_i, \overline{v}_i] \subset \mathbb{R}_+$. We assume that $v_i(\theta)$ is continuous in θ and nondecreasing in θ_i.

Players compete by simultaneously choosing the amount of effort they invest. Each player i chooses $x_i \in \mathbb{R}_+$. The probability that i wins the contest (or, alternatively, the share of the prize i obtains) depends on the vector of all efforts $x := (x_1, \dots, x_n)$. This winning probability is given by the CSF $p_i : \mathbb{R}^n_+ \to [0,1]$, where $p_i(x)$ is continuous in x and nondecreasing in x_i.

Investing effort is costly. Player i's cost of providing effort x_i can be described by a continuous function $c_i : \mathbb{R}_+ \to \mathbb{R}_+$. We assume that $c_i(0) = 0$ and that there exists an $\overline{x}_i \in \mathbb{R}_+$ such that $c_i(x_i) > \overline{v}_i$ for all $x_i > \overline{x}_i$.

Let player i's ex post payoff be denoted by

$$u_i(x, \theta) := p_i(x) v_i(\theta) - c_i(x_i).$$

Suppose each of player i's opponents j uses a pure strategy $\xi_j : [\underline{\theta}_j, \overline{\theta}_j] \to \mathbb{R}_+$ and let θ_{-i} be the vector of the signals of i's opponents. To simplify the notation we will use $(x_i, \xi_{-i}(\theta_{-i}))$ to refer to the vector $(\xi_1(\theta_1), \dots, \xi_{i-1}(\theta_{i-1}), x_i, \xi_{i+1}(\theta_{i+1}), \dots, \xi_n(\theta_n))$. Then, i's objective function, i.e., i's interim expected payoff amounts to

$$\begin{aligned} U_i(x_i, \theta_i) :=& E\left[u_i\big((x_i, \xi_{-i}(\theta_{-i})), \theta\big) \,\middle|\, \theta_i \right] \\ =& \int_{\Theta_{-i}} p_i\big(x_i, \xi_{-i}(\theta_{-i})\big) v_i(\theta) \prod_{j \neq i} f_j(\theta_j) d\theta_{-i} - c_i(x_i) \end{aligned} \tag{1}$$

where $\Theta_{-i} := [\underline{\theta}_1, \overline{\theta}_1] \times \cdots \times [\underline{\theta}_{i-1}, \overline{\theta}_{i-1}] \times [\underline{\theta}_{i+1}, \overline{\theta}_{i+1}] \times \cdots \times [\underline{\theta}_n, \overline{\theta}_n]$. We are now ready to prove the main result by making use of Athey (2001).

Theorem 1. *There is a Bayesian Nash equilibrium where each player i uses a nondecreasing pure strategy $\xi_i(\theta_i)$.*

Proof. In order to prove the result we apply Corollary 2.1 in Athey (2001). In the following, we will verify that our model meets all the requirements of Athey's corollary. First note that our assumptions are consistent with Athey's assumption A1: Each density f_i is bounded and atomless and the integral in (1) exists since u_i is continuous in all of its arguments.

Because $p_i(x)$ is nondecreasing in x_i and $v_i(\theta)$ is nondecreasing in θ_i,

$$\Big(p_i(x_i^H, x_{-i}) - p_i(x_i^L, x_{-i})\Big)\, v_i(\theta_i^L, \theta_{-i}) - \Big(c_i(x_i^H) - c_i(x_i^L)\Big) \ \geq (>)\ 0$$
$$\Rightarrow\ \Big(p_i(x_i^H, x_{-i}) - p_i(x_i^L, x_{-i})\Big)\, v_i(\theta_i^H, \theta_{-i}) - \Big(c_i(x_i^H) - c_i(x_i^L)\Big) \ \geq (>)\ 0$$

for all $x_i^H > x_i^L$ and $\theta_i^H > \theta_i^L$. As the signals θ are independent, the above continues to hold when replacing x_{-i} by $\xi_{-i}(\theta_{-i})$ and taking the expectation with respect to θ_{-i}. Consequently, for all $x_i^H > x_i^L$ and $\theta_i^H > \theta_i^L$,

$$U_i(x_i^H, \theta_i^L) - U_i(x_i^L, \theta_i^L) \ \geq (>)\ 0 \quad \Rightarrow \quad U(x_i^H, \theta_i^H) - U(x_i^L, \theta_i^H) \ \geq (>)\ 0,$$

i.e., each player i's objective function $U_i(x_i, \theta_i)$ satisfies single crossing of incremental returns in (x_i, θ_i). Therefore, Athey's *Single Crossing Condition for games of incomplete information* is satisfied.

Now, observe that $u_i(x,\theta) \leq v_i(\theta) - c_i(x_i) \leq \overline{v}_i - c_i(x_i)$. For $x_i > \overline{x}_i$ we have $u_i(x,\theta) < 0$ while with $x_i = 0$ a player can guarantee himself $u_i(x,\theta) \geq 0$. Effort levels $x_i > \overline{x}_i$ are clearly dominated. Thus, we can restrict players' actions to $x_i \in [0, \overline{x}_i]$ for all i – a closed interval as assumed by Athey. Moreover, $u_i(x,\theta)$ is continuous in x for all i. Existence of a pure-strategy Bayesian Nash equilibrium in nondecreasing strategies hence directly follows from Corollary 2.1 in Athey (2001). □

3 Discussion

Our existence result applies to private information versions of many contests considered in the literature. Under complete information, much attention is devoted to contests with a CSF that takes the form

$$p_i(x) = \frac{g_i(x_i)}{\sum_{j=1}^{n} g_j(x_j)} \tag{2}$$

where each $g_i(x_i)$ is a continuous and increasing function. Most importantly, Skaperdas (1996, Theorem 1) provides an axiomatization for the symmetric case where $g_i = g$ for all i. Under the condition that $g_i(0) > 0$, Theorem 1 implies existence of a pure-strategy equilibrium if such a contest is held among privately informed contestants.[3] A related class of contests where Theorem 1 can also be applied to features the CSF

$$p_i(x) = \frac{h_i(x_i)}{1 + \sum_{j=1}^{n} h_j(x_j)}$$

where each $h_i(x_i)$ is continuous and strictly increasing. This CSF is, e.g., considered by Dasgupta and Nti (1998) and an axiomatization is provided by Blavatskyy (2010).

Under complete information contests with a CSF as in (2) are known to have only mixed-strategy equilibria if g_i is too convex (e.g., Cornes and Hartley, 2005). In contrast, Theorem 1 shows that with privately informed players there is always a pure-strategy equilibrium. This echoes a well-known property of the perfectly discriminating all-pay auction, having a pure-strategy equilibrium under incomplete but not under complete information. Indeed, the all-pay auction can be obtained as the limit of a sequence of contests with increasingly convex g_i. Although not needed for equilibrium existence under incomplete information, assuming g_i to be concave usually simplifies the analysis, especially since it allows for the equilibrium strategies to be

[3] A popular specification, proposed by Tullock (1980), is $g(x) = x^R$ for some $R > 0$ and $p_i(0, \ldots, 0) = \frac{1}{n}$. Theorem 1 does not directly apply because this CSF exhibits a discontinuity at $x = (0, \ldots, 0)$. However, if each player's action is restricted to a finite set, Theorem 1 in Athey (2001) implies existence of a pure-strategy equilibrium. Moreover, equilibrium existence with continuous actions is studied by Fey (2008) and Ryvkin (2010).

characterized by first-order conditions.[4]

Hirshleifer (1989) proposes a symmetric specification of (2) where $g(x_i) = \exp(kx_i)$ with $k > 0$. If $n = 2$, this is at the same time also a special case of the class of contests introduced by Lazear and Rosen (1981). They assume that the player i with the highest $x_i + \varepsilon_i$ wins the contest, where ε_i is a randomly distributed noise term realized only after efforts have been chosen. Let L_i denote the distribution function of $\varepsilon_j - \varepsilon_i$ for $i, j = 1, 2$ and $i \neq j$. The corresponding CSF hence becomes $p_i(x) = L_i(x_i - x_j)$. For continuous L_i Theorem 1 ensures existence of a pure-strategy equilibrium if players have private information.

Under complete information Che and Gale (2000) analyze a family of piecewise linear L_i and find that pure-strategy equilibria disappear if the contest is similar enough to the all-pay auction. According to Theorem 1, also for this class of contests existence of a pure-strategy equilibrium is restored when introducing private information.

Establishing the existence of a monotone pure-strategy equilibrium, we make a first step towards a better understanding of imperfectly discriminating contests with private information. Studying the properties of such equilibria seems to be a promising direction for future research.

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[4]Equilibrium strategies can, in general, not be obtained in closed form. For a symmetric version of (2) where g is an affine function Wasser (2012) uses first-order conditions to derive bounds for expected efforts and to approximate the equilibrium numerically.

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