Extrinsic Uncertainty Revisited: Variable Capital Utilization and Returns to Scale

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Abstract

This paper presents a one-sector optimal growth model with variable capacity services and production externalities. It uses a new formulation of the endogenous capital utilization rate in which utilization costs appear in the form of variable maintenance expenses. I find that indeterminacy arises at approximate constant returns to scale. This result challenges the viewpoint that indeterminacy is empirically implausible.

1 Introduction

Much of the current discussion of indeterminacy in dynamic general equilibrium is based on the assumption of market imperfection for example stemming from increasing returns to scale in production. These market defects make the economy vulnerable to a pernicious type of event known as sunspots. Sunspots represent purely extraneous information which may affect economic variables only because the public believes it does. These beliefs, unrelated to fundamentals, can induce the economy to undergo fluctuations. However, the empirical support for significant increasing returns that are needed to generate indeterminacy is not strong.[1] For instance, Basu and Fernald (1997) suggest that returns to scale are close to constant. Laitner and Stolyarov (2004) suggest point estimates ranging from 1.09 to 1.11.

The current essay presents a dynamic one-sector representative agent economy which requires insignificant increasing returns to scale compatible with indeterminacy. The model incorporates a new formulation of endogenous capital utilization taken from the sticky-price model by Christiano, Eichenbaum and Evans (2001). Christiano et al. (2001) promote a theory in which costs that occur from increasing the utilization rate arise in terms of a direct loss of output. One may think of these costs as maintenance expenses which no longer make available some production for consumption and investment purposes. Alternatively, the costs may represent resources (physical output or time) used in reorganizing the production process when its intensity is boosted. I am able to show that with variable capital utilization the level of returns to scale needed for indeterminacy may be reduced dramatically: the minimum scale economies are essentially constant and amount to less than 1.003.

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[1] See Farmer and Benhabib (1999) for a recent survey on indeterminacy. Certainly, in multisector-models indeterminacy is possible at small increasing returns, yet that model class typically fails to replicate a number of important stylized (business cycle) facts.

The model is related to Wen (1998) who also extends the one-sector model by variable capacity utilization. In that economy, more intense capital services result in a larger depreciation of the capital stock. Wen demonstrates that indeterminacy can be obtained at increasing returns to scale of 1.108. In contrast, Benhabib and Farmer's (1994) one-sector model with constant capital utilization requires increasing returns of 1.43 or higher for indeterminacy which is clearly outside the empirically plausible range.

2 The model

Let us assume that the representative agent's preferences depend on consumption, $c_{i,t}$, and labor, $l_{i,t}$, and, furthermore, the individual's infinite-horizon utility may be written

$$U = \int_0^\infty u\left(c_{i,t}, l_{i,t}\right) e^{-\rho t} dt \tag{1}$$

where ρ is the nonnegative subjective rate of time preference. I will concentrate on instantaneous utility being additive separable in consumption and labor

$$u\left(c_{i,t}, l_{i,t}\right) = \ln c_{i,t} - \psi l_{i,t} \qquad \psi > 0.$$

The reason for frequently restricting the study to the case of logarithmic utility is that it is the only additively separable utility function consistent with balanced growth. Indivisible labor is standard in the real business cycles literature. If capital is subject to evaporative decay at constant rate δ, then growth of the capital stock, $k_{i,t}$, is specified by the differential equation

$$\frac{dk_{i,t}}{dt} = x_{i,t} - \delta k_{i,t} \qquad \delta > 0 \tag{2}$$

where $x_{i,t}$ denotes the household's investment expenditures.

The economy as a whole is affected by organizational synergies that cause the output of an individual unit to be higher if all other units in the economy are producing more. The production complementarities are taken as given for the individual optimizer and they cannot be priced or traded. Hence, technology can be expressed by

$$y_{i,t} = A_t^{\gamma}(u_{i,t}k_{i,t})^{\alpha} l_{i,t}^{1-\alpha} \qquad \alpha \in (0,1) \tag{3}$$

where $y_{i,t}$ is output and $u_{i,t}$ is the non-constant rate of capital services. The term A_t denotes aggregate externalities

$$A_t = (u_t k_t)^{\alpha} l_t^{1-\alpha}.$$

The variables k_t, u_t and l_t refer to economy-wide averages. Returns to scale in production are measured by the parameter γ. Finally, the agent's income identity is

$$y_{i,t} = c_{i,t} + x_{i,t} + a(u_{i,t})k_{i,t} \qquad a' > 0, \quad a'' > 0. \tag{4}$$

The function $a(u_{i,t})$ stands for the costs of setting the utilization rate. The fact that this function is assumed to be convex guarantees that the agent's maximization problem is concave and has a unique solution. The formulation follows Christiano et al. (2001). Costs that occur from increasing the utilization rate arise in terms of a direct loss of available output. Intensifying capital services results in a higher output (the left hand side of equation (4)) but it also reduces the amount of output that is available for consumption and investment purposes (the right hand side of equation (4)). In addition to the constant capital depreciation rate, the cost formulation is the main difference to the Wen (1998) indeterminacy economy.

Given the assumptions on (explicit and implicit) functional forms, $u\left(c_{i,t}, l_{i,t}\right)$ and $y_{i,t} - c_{i,t} - \delta k_{i,t} - a(u_{i,t})k_{i,t}$ are (strictly) concave in consumption, labor, capital and the utilization rate. The Hamiltonian for the problem may be defined as

$$\mathcal{H} = u\left(c_{i,t}, l_{i,t}\right) + \lambda_t \left(A_t^{\gamma}(u_{i,t}k_{i,t})^{\alpha} l_{i,t}^{1-\alpha} - c_{i,t} - (a(u_{i,t}) + \delta)k_{i,t}\right)$$

where λ_t is the multiplier associated with the state variable $k_{i,t}$.

3 Dynamics

Since all agents are identical, the economy-wide averages of variables must be equal to the corresponding values of the individual economic units in symmetric equilibrium. Then, the dynamics are given by the static conditions

$$\lambda_t = \frac{1}{c_t} \tag{5}$$

$$\psi l_t = (1-\alpha)\frac{y_t}{c_t} \tag{6}$$

$$a'(u_t)u_t = \alpha\frac{y_t}{k_t} \tag{7}$$

$$c_t + x_t + a(u_t)k_t = y_t = (u_t k_t)^{\alpha(1+\gamma)} l_t^{(1-\alpha)(1+\gamma)}, \tag{8}$$

the dynamic relations

$$\frac{dk_t}{dt} = x_t - \delta k_t \tag{9}$$

$$\frac{d\lambda_t/dt}{\lambda_t} = \delta + \rho + a(u_t) - \alpha y_t k_t^{-1}, \tag{10}$$

and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\frac{k_t}{c_t} = 0.$$

Let us formulate the following definition.

Definition 1 (Perfect foresight equilibrium) *In the artificial economy, a perfect foresight equilibrium is a sequence* $(k_t,\ \lambda_t)$ *and an initial capital stock* $k(0) > 0$ *satisfying (5) to (10) and the transversality condition.*

As in Christiano et al. (2001), I normalize $u = 1$ and set $a(u) = 0$ in steady state. Thus, the costs do not occur when the economy is in its stationary state – one therefore may think of the steady state representing normal economic times and additional costs show up once output is boosted beyond that level. The unique steady state implies[2]

$$\rho + \delta = \alpha \frac{y}{k} = a'(u)$$

$$\delta = \frac{x}{k}$$

and

$$1 = \frac{c}{y} + \frac{x}{y} = \frac{c}{y} + \delta \frac{k}{y}.$$

The model does not have a closed form solution. Thus, I derive the local dynamics of the model by taking a Taylor Series approximating around the unique steady state. The steady state conditions involve all terms needed to conduct the linearization (the Appendix presents the complete model). The exception comes with the linear version of equation (7)

$$\widehat{k}_t = \widehat{y}_t - (1 + \sigma_a)\widehat{u}_t$$

which entails the term

$$\sigma_a \equiv \frac{a''(u)u}{a'(u)} > 0.$$

This cost function elasticity originates from a three parameter class yet only u and $a'(u)$ are determined from the set of steady state conditions. Thus, σ_a remains a free parameter. The model's local dynamics boil down to the matrix differential equation

$$\begin{pmatrix} d \ln \lambda_t / dt \\ d \ln k_t / dt \end{pmatrix} = \mathbf{J} \begin{pmatrix} \ln \lambda_t - \ln \lambda \\ \ln k_t - \ln k \end{pmatrix} \tag{11}$$

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[2] The alert reader will note that the assumptions regarding the cost function imply that below the steady state the function becomes a benefit function.The Appendix discusses the issue.

where $\mathbf{J}$ denotes the 2×2 Jacobian matrix of partial derivatives. Indeterminacy is defined as follows.

Definition 2 (Indeterminacy of steady state) *The equilibrium is indeterminate if there exists an infinite number of perfect foresight equilibrium sequences.*

The co-state variable λ_t is a jump variable and the capital stock is a state variable. Thus, indeterminacy of the linear system (11) requires that both eigenvalues of $\mathbf{J}$ have negative real parts; the steady state is a sink. Since the trace of the matrix is the sum of its eigenvalues and the determinant is the product of the eigenvalues, indeterminacy can be restated as

$$\mathrm{Tr}\mathbf{J} < 0 < \mathrm{Det}\mathbf{J}.$$

Similarly, the steady state is saddle path stable if $\mathrm{Det}\mathbf{J} < \mathbf{0}$ and it is unstable (a source) if

$$\mathrm{Tr}\mathbf{J} > \mathbf{0} \quad \text{and} \quad \mathrm{Det}\mathbf{J} > \mathbf{0}.$$

Translated into (11), indeterminacy implies that, following purely extraneous information, λ_t may jump at $t = 0$ to any of an infinite number of stable equilibrium trajectories.

Throughout the remainder of the essay, I will assume that the following holds:

Assumption *The level of returns to scale arising from the externalities is always nondecreasing:* $\gamma \geq 0$. *However, the level of increasing returns from all sources is modest:* $\alpha(1+\gamma) < 1$.

The assumption includes values of the externality that are empirically plausible given evidence in Basu and Fernald (1997) and others. For example, Burnside, Eichenbaum and Rebelo (1995) suggest that when variable utilization is considered as in the present model, the evidence for increasing returns is weak. They report a point estimate of 0.98, however, their standard error of 0.34 is large. The assumption also implies that increasing returns are not high enough to induce endogenous growth.

Let us now focus on the analysis of indeterminacy in this economy. Let us start by checking the determinant of **J**

$$\mathrm{Det}\mathbf{J} = \frac{\overbrace{\sigma_a\left(\rho+\delta\right)\left(\rho-(\alpha-1)\delta\right)(\alpha(1+\gamma)-1)}^{(-)}}{\alpha\left(\alpha\sigma_a+\gamma(\sigma_a\alpha-1)-\gamma\right)}$$

to derive necessary conditions. In the absence of externalities ($\gamma = 0$), the equilibrium is unique since the determinant implies one positive eigenvalue

$$\mathrm{Det}\mathbf{J} = -\frac{(1-\alpha)\left(\rho+\delta\right)\left(\rho-(\alpha-1)\delta\right)}{\alpha^2} < 0.$$

The minimum extent of increasing returns to scale that is required to yield steady states other than a saddle is given by

$$\gamma^{\min} = \frac{\alpha\sigma_a}{1-\sigma_a(\alpha-1)} > 0$$

which means that the denominator of Det**J** becomes negative. It is clear that $\gamma^{\min}$ is an increasing function in σ_a; it approaches zero as $\sigma_a \rightarrow 0$ and it becomes $\frac{\alpha}{1-\alpha}$ as $\sigma_a \rightarrow \infty$. This upper level is the same as found in Harrison and Weder (2002, Proposition 1) and indicates that the current model nests the Benhabib and Farmer (1994) economy as a special case. Economically, it implies that if adjusting capital utilization becomes more costly the model needs a higher degree of externalities while still obtaining indeterminacy (or a source). If the costs approach infinity, the capital owners will find it optimal to keep the rate of utilization constant and the model behaves exactly as Benhabib and Farmer (1994).

A further condition for indeterminacy is that the trace of **J**

$$\mathrm{Tr}\mathbf{J} = \frac{\alpha\rho\sigma_a+\gamma\delta\sigma_a+\rho\gamma(\alpha\sigma_a-1)}{\alpha\sigma_a+\gamma(\sigma_a\alpha-1)-\gamma}$$

is negative. One readily sees that the trace's denominator is negative whenever Det$\mathbf{J}$ is positive. Thus, indeterminacy requires that the denominator of Tr$\mathbf{J}$ is positive. In fact, for all non-negative γ, the trace is always negative when

$$\sigma_a > \frac{\rho}{\alpha\rho + \delta}.$$

Otherwise, a necessary relation between minimum values of σ_a and maximum increasing returns for which the trace remains negative is

$$\gamma < \frac{\alpha\rho\sigma_a}{\rho - \sigma_a(\alpha\rho + \delta)}.$$

Again, it is clear that the maximum value of γ is an increasing function in σ_a; it approaches zero as $\sigma_a \to 0$.

Note that, for $\sigma_a < \frac{\rho}{\alpha\rho+\delta}$, the two boundaries that enclose the indeterminacy region satisfy

$$\gamma^{\max} = \frac{\alpha}{\frac{1}{\sigma_a} - \alpha - \frac{\delta}{\rho}} > \frac{\alpha}{\frac{1}{\sigma_a} - \alpha + 1} = \gamma^{\min}$$

thus, for every admissible σ_a there exists a γ for which indeterminacy obtains. Moreover, one can show that there exists a threshold level at which the steady state changes from sink to source, the eigenvalues are purely imaginary. This suggests the possibility of Hopf bifurcations in which any trajectory that diverges away from the completely unstable stationary state settles down to limit cycles or to some more complicated attracting sets. Furthermore, at $\gamma = (1 - \alpha)/\alpha$ the determinant of $\mathbf{J}$ is zero going from positive to negative as returns to scale increase: Thus, even larger externalities than those considered do not yield indeterminacy.

To understand the economic mechanism that creates the continuum of solutions, let us walk through a simple example. Suppose that at $t = 0$, and in the presence of returns to scale and upon increasingly optimistic expectations, (say, an anticipated higher return to capital) the household will raise investment. This shifts the labor supply curve outwards with the effect of higher employment. In the model, this leads to higher capital utilization rates and an outward shift of labor demand. This allows today's consumption to be expanded as well. In any case, given sufficient increasing returns, the household will find itself with an augmented future capital stock as well as higher output, capital utilization, hours and labor productivity: its initial optimistic expectations are self-fulfilled. Yet, for this sunspot movement to be stationary, the economy must move back to its steady state. Increasing output drives up utilization costs which ultimately bring the expansion to a halt. If adjusting capital utilization becomes very costly – the parameter σ_a is large – then one needs a higher degree of externalities while still obtaining indeterminacy. However, when

$$\sigma_a < \frac{\rho}{\alpha\rho + \delta}$$

then for any magnitude of adjustment costs σ_a, one can find returns to scale that are too large in the sense of the steady state becoming a sink. This means that after some (small) pertubation away from the steady state equilibrium, the endogenous dynamics do not generate a stable trajectory that brings the economy ultimately back to its starting position; equilibria are nonstationary. Yet, if the above inequality is not fulfilled, the trace can never become positive in the admissible parameter space and it appears that the original Benhabib and Farmer (1994, $\sigma_a \to \infty$) economy does not allow Hopf bifurcations.[3]

Finally, it is worthwhile to look at particular numerical parameter constellations that imply indeterminacy. Calibration is now routine in a wide range of macroeconomic areas. Table 1 summarizes the calibration of the model's deep parameters. It is the same as in Wen (1998). The fundamental period in the model is one quarter. The value for α is chosen such that the capital share amounts to thirty percent and the quarterly rate of capital de-

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[3] Recall that the determinant becomes negative for $\gamma > (1-\alpha)/\alpha$. Coury and Wen (2001) find a related result in a discrete-time version of the Benhabib and Farmer (1994) model.

preciation is 2.5 percent. The subjective rate of time preference equals one percent.

Table 1		
α	δ	ρ
0.30	0.025	0.01

Two parameters remain uncalibrated in the linearized model: the degree of increasing returns and the elasticity of the utilization cost function σ_a. Christiano et al. (2001) do not calibrate the elasticity σ_a but estimate it from data. In particular, they compute σ_a by minimizing a distance measure between their model and empirical impulse response functions. They advocate a value of $\sigma_a = 0.01$. If we follow Christiano et al. (2001), then indeterminacy arises at $\gamma^{\min} = 0.00297915$ and the model turns into a source at $\gamma^{\max} = 0.00308642$. Increasing utilization costs to, say $\sigma_a = 0.1$, raises $\gamma^{\min}$ to 0.0280374 and $\gamma^{\max} = 0.0416667$. Moreover, by letting σ_a approach infinity (i.e. the model collapses into the Benhabib-Farmer, 1994, economy with a constant utilization rate) the minimum returns to scale become 1.428571. Overall, in the present artificial economy increasing returns that are consistent with indeterminacy are significantly lower than those found in other one-sector models, and, more importantly, the degree of scale economies is empirically plausible.

Why does indeterminacy arise at lower increasing returns than in other one-sector models? First of all, note that after a positive demand shock the reaction of the economy must be such to induce a higher return to capital. With constant capital services, large increasing returns are needed to produce such an upward trend of capital returns (Benhabib and Farmer, 1994). The insight from Wen (1998) was that by making utilization variable, the same effect occurs at lower increasing returns since a rise of u_t shifts the capital returns schedule outwards endogenously.

What is the reason for the present economy's need for much lower increasing returns than Wen's? Both model arrangements consist of costs that check the service rate from going to infinity. In Wen (1998) this is done by making the rate of capital depreciation variable. The effect is reducing the future capital stock as well as future output and thereby working against the indeterminacy mechanism. The present model operates via contemporaneous output losses. Again, the mechanism is counteracting the indeterminacy mechanism. Wen (1998) claims that his model does not nest the Benhabib and Farmer (1994) economy which is due to his restrictive assumptions on the depreciation function. However, it is easily shown that the present model and Wen's are observationally equivalent. That is, the relevant conditions imply (the second lines are taken from Wen, 1998)

$$\begin{aligned} dk_t/dt &= y_t - c_t - (a(u_t) + \delta)k_t \\ dk_t/dt &= y_t - c_t - \frac{1}{\theta}u_t^\theta k_t \qquad \theta > 0 \end{aligned}$$

and

$$\begin{aligned} a'(u_t)u_t &= \alpha\frac{y_t}{k_t} \\ u_t^\theta &= \alpha\frac{y_t}{k_t}. \end{aligned} \tag{12}$$

Wen's formulation does not allow to calibrate θ – in fact, this parameter is pinned down by first-order conditions and the l.h.s. of (12) becomes a relatively "steep function" which pushes up the minimum increasing returns.[4] In an important sense, the Christiano et al. (2001) definition of variable capital utilization introduces a degree of freedom that enables to calibrate σ_a from empirical studies. Phrased alternatively, answering the above question boils down to which of the counteracting effects is weaker. I find that for low values of σ_a the current model obtains indeterminacy at smaller returns to scale than Wen (1998). When σ_a rises to infinity, the economy approaches

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[4] Given Table 1's calibration the minimum returns to scale are $\gamma^{\min} = 0.09375001$ in a continuous-time version of the Wen model.

Benhabib-Farmer (1994) and minimum returns to scale become implausibly large; in effect, in this case the minimum returns are larger than in Wen (1998). However, Christiano et al. (2001) suggest that σ_a is in fact close to zero, hence, the present indeterminacy-economy's need for only insignificant increasing returns.

4 Conclusion

This paper has presented a one-sector business cycle model with variable capacity utilization and externalities that come from aggregate economic activity. It uses a new formulation of the endogenous capital utilization rate in which utilization costs show up as maintenance expenses which no longer make available some produced output for consumption and investment purposes. I find that indeterminacy arises at approximate constant returns to scale. The result challenges the viewpoint that indeterminacy is empirically implausible in real models.

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5 Appendix

Log-linearizing yields the following static equations (denote $\ln x_t - \ln x$ by $\widehat{x}_t$ *et cetera*)

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\begin{aligned}
0 &= \widehat{y}_t - \widehat{l}_t - \widehat{c}_t \\
\alpha(1+\gamma)\widehat{k}_t &= \widehat{y}_t - \alpha(1+\gamma)\widehat{u}_t - (1-\alpha)(1+\gamma)\widehat{l}_t \\
\widehat{k}_t &= \widehat{y}_t - (1+\sigma_a)\widehat{u}_t \\
0 &= \widehat{y}_t - \alpha\widehat{u}_t - \frac{x}{y}\widehat{x}_t - \frac{c}{y}\widehat{c}_t \\
\widehat{\lambda}_t &= -\widehat{c}_t .
\end{aligned}
$$

The two dynamic equations are

$$\begin{aligned}d\ln\lambda_t/dt = (\rho+\delta)\widehat{k}_t - (\rho+\delta)\widehat{y}_t + (\rho+\delta)\widehat{u}_t \\ d\ln k_t/dt = -\delta\widehat{k}_t + \delta\widehat{x}_t.\end{aligned}$$

The static equations can be combined to

$$\Pi_1\begin{pmatrix}\widehat{\lambda}_t\\ \widehat{k}_t\end{pmatrix} = \Pi_2\begin{pmatrix}\widehat{y}_t\\ \widehat{u}_t\\ \widehat{x}_t\\ \widehat{l}_t\\ \widehat{c}_t\end{pmatrix}$$

and the dynamic equations yield

$$\begin{pmatrix}d\ln\lambda_t/dt\\ d\ln k_t/dt\end{pmatrix} = \mathbf{J}_1\begin{pmatrix}\widehat{\lambda}_t\\ \widehat{k}_t\end{pmatrix} + \mathbf{J}_2\begin{pmatrix}\widehat{y}_t\\ \widehat{u}_t\\ \widehat{x}_t\\ \widehat{l}_t\\ \widehat{c}_t\end{pmatrix}.$$

The model reduces to (11) with the Jacobian

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2\Pi_2^{-1}\Pi_1.$$

The present model employs a general cost function of capital utilization. Wen (1998) proceeds along a different way in calibrating endogenous capital utilization: he picks utilization costs via capital depreciation

$$\delta_t = \frac{1}{\theta}u_t^{\theta}.$$

Wen uses the first-order conditions for the choice of steady state to determine θ. The disadvantage is that it remains unclear what empirical fact the endogenously determined parameter is fitting – in fact, the empirical work in Christiano et al. (2001) and others suggest that the costs may be small.

The alert reader will note that the assumptions regarding the cost function, $a(u_t)$,imply that below the steady state the function becomes a benefit function. Replacing the formulation by $\tilde{a}(u) = \zeta + a(u)$, $\zeta > 0$, is a natural alternative. In this case, the steady state involves

$$\delta = \frac{x}{k} = \frac{x}{y}\frac{\rho + \delta}{\alpha}$$

and

$$\frac{c}{y} = 1 - \frac{x}{y} - \frac{\zeta}{y} = 1 - \frac{x}{y} - \tilde{\zeta}.$$

The additional term, $\tilde{\zeta}$, does only appear in the determinant of $\mathbf{J}$:

$$\mathrm{Det}\mathbf{J} = \frac{\overbrace{\sigma_a \left(\rho + \delta\right) \left(\rho - (\alpha - 1)\delta\right)(\alpha(1 + \gamma) - 1)}^{(-)} \; \tilde{\zeta}(\rho + \delta)\gamma}{\alpha \left(\alpha\sigma_a + \gamma(\sigma_a\alpha - 1) - \gamma\right)}$$

As is straightforward to see, the sign of the determinant remains unaffected.